

STANDARD PROTOCOL COMPLEXES FOR THE IMMEDIATE SNAPSHOT READ/WRITE MODEL

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ABSTRACT. In this paper we consider a family of abstract simplicial complexes which we call immediate snapshot complexes. Their definition is motivated by theoretical distributed computing. Specifically, these complexes appear as protocol complexes in the general immediate snapshot execution model.

In order to define and to analyze the immediate snapshot complexes we use the novel language of witness structures. We develop the rigorous mathematical theory of witness structures and use it to prove several combinatorial as well as topological properties of the immediate snapshot complexes. In particular, we prove that these complexes are simplicially homeomorphic to simplices.

1. THE MOTIVATION FOR THE STUDY OF IMMEDIATE SNAPSHOT PROTOCOL COMPLEXES

One of the core computational models, which is used to understand the shared-memory communication between a finite number of processes is the so-called *immediate snapshot read/write model*. In this model, a number of processes are set to communicate by means of a shared memory. Each process has an assigned register, and each process can perform two types of operations: *write* and *snapshot read*. The write operation simply writes the entire state of the process into its assigned register; the snapshot read operation reads the entire memory in one atomic step. The order in which a process performs these operations is controlled by the distributed protocol, whose execution is asynchronous, satisfying an additional condition. Namely, we assume that at each step a group of processes gets active. First this group simultaneously writes its values to the memory, then it simultaneously performs a snapshot read. This way, each execution can be encoded by a sequence of groups of processors which become active at each turn. More details on this computational model, the associated protocol complexes and its equivalence with other models can be found in a recent book [HKR].

In this paper we consider the distributed protocols for $n + 1$ processes indexed $0, \dots, n$, where the protocol of k -th processor says to run r_k rounds and then to stop. Let the associated protocol complex be called $P(r_0, \dots, r_n)$. Our first contribution is to give a rigorous purely combinatorial definition of $P(r_0, \dots, r_n)$. To do this, we introduce new mathematical objects, which we call *witness structures* and use them as a language to define and to analyze this family of simplicial complexes. The special case $r_0 = \dots = r_n = 1$ corresponds to the so-called *standard chromatic subdivision* of a simplex, see [Ko12, Ko13], the cases where some $r_i \geq 2$ are new.

Key words and phrases. collapses, distributed computing, combinatorial algebraic topology, immediate snapshot, protocol complexes.

The simplicial complexes $P(r_0, \dots, r_n)$ are of utmost importance in the shared-memory communication. We perform a thorough analysis of their combinatorial and topological structure. Our main tool is the canonical decomposition of $P(r_0, \dots, r_n)$, with strata corresponding to various groups of processes which take the first turn of the computation. Our main theorem states that each simplicial complex $P(r_0, \dots, r_n)$ is simplicially homeomorphic to an n -simplex.

2. THE LANGUAGE OF WITNESS STRUCTURES

2.1. Some notations. We let \mathbb{Z}_+ denote the set of nonnegative integers $\{0, 1, 2, \dots\}$. For a natural number n we shall use $[n]$ to denote the set $\{0, \dots, n\}$.

For a finite subset $S \in \mathbb{Z}_+$, such that $|S| \geq 2$, we let $\text{smax } S$ denote the *second* largest element $\max(S \setminus \{\max S\})$. For a family of finite sets $(S_i)_{i \in I}$, $S_i \subset \mathbb{Z}_+$, we let $\max(S_i)_{i \in I}$ be the short-hand notation for $\max_{i \in I}(\max S_i) = \max(\bigcup_{i \in I} S_i)$.

For a set S and an element a , we set

$$\chi(a, S) := \begin{cases} 1, & \text{if } a \in S; \\ 0, & \text{otherwise.} \end{cases}$$

Whenever $(X_i)_{i=1}^t$ is a family of topological spaces, we set $X_I := \bigcup_{i \in I} X_i$. Also, when no confusion arises, we identify one-element sets with that element, and write, e.g., p instead of $\{p\}$.

We shall use \hookrightarrow to denote simplicial inclusions, \sim to denote simplicial isomorphisms, and we use $\xrightarrow{\cong}$, and $\xrightarrow{\cong}$, to denote homeomorphisms.

2.2. Round counters. Our main objects of study, the immediate snapshot complexes, are indexed by finite tuples of nonnegative integers. We need to be more specific about the formalism of this indexing set.

Definition 2.1. *Given a function $\bar{r} : \mathbb{Z}_+ \rightarrow \mathbb{Z}_+ \cup \{\perp\}$, we consider the set*

$$\text{supp } \bar{r} := \{i \in \mathbb{Z}_+ \mid \bar{r}(i) \neq \perp\}.$$

*This set is called the **support set** of \bar{r} .*

*A **round counter** is a function $\bar{r} : \mathbb{Z}_+ \rightarrow \mathbb{Z}_+ \cup \{\perp\}$ with a finite support set.*

Obviously, a round counter can be thought of as an infinite sequence $\bar{r} = (\bar{r}(0), \bar{r}(1), \dots)$, where, for all $i \in \mathbb{Z}_+$, either $\bar{r}(i)$ is a nonnegative integer, or $\bar{r}(i) = \perp$, such that only finitely many entries of \bar{r} are nonnegative integers. We shall frequently use a short-hand notation $\bar{r} = (r_0, \dots, r_n)$ to denote the round counter given by

$$\bar{r}(i) = \begin{cases} r_i, & \text{for } 0 \leq i \leq n; \\ \perp, & \text{for } i > n. \end{cases}$$

Definition 2.2. *Given a round counter \bar{r} , the number $\sum_{i \in \text{supp } \bar{r}} \bar{r}(i)$ is called the **cardinality** of \bar{r} , and is denoted $|\bar{r}|$. The sets*

$$\text{act } \bar{r} := \{i \in \text{supp } \bar{r} \mid \bar{r}(i) \geq 1\} \text{ and } \text{pass } \bar{r} := \{i \in \text{supp } \bar{r} \mid \bar{r}(i) = 0\}$$

*are called the **active** and the **passive** sets of \bar{r} .*

Definition 2.3. For an arbitrary pair of disjoint finite sets $A, B \subseteq \mathbb{Z}_+$ we define a round counter $\chi_{A,B}$ given by

$$\chi_{A,B}(i) := \begin{cases} 1, & \text{if } i \in A; \\ 0, & \text{if } i \in B. \end{cases}$$

Furthermore, for an arbitrary round counter \bar{r} , we set $\chi(\bar{r}) := \chi_{\text{act } \bar{r}, \text{pass } \bar{r}}$.

We note that $\text{supp } \bar{r} = \text{supp } (\chi(\bar{r}))$. In the paper we shall also the short-hand notation $\chi_A := \chi_{A, \emptyset}$.

We define two operations on the round counters. To start with, assume \bar{r} is a round counter and we have a subset $A \subseteq \mathbb{Z}_+$. We let $\bar{r} \setminus A$ denote the round counter defined by

$$(\bar{r} \setminus A)(i) = \begin{cases} \bar{r}(i), & \text{if } i \notin A; \\ \perp, & \text{if } i \in A. \end{cases}$$

We say that the round counter $\bar{r} \setminus A$ is obtained from \bar{r} by the *deletion* of A . Note that $\text{supp } (\bar{r} \setminus A) = \text{supp } \bar{r} \setminus A$, $\text{act } (\bar{r} \setminus A) = \text{act } \bar{r} \setminus A$, and $\text{pass } (\bar{r} \setminus A) = \text{pass } \bar{r} \setminus A$. Furthermore, we have $\chi(\bar{r} \setminus A) = \chi(\bar{r}) \setminus A$. Finally, we note for future reference that for $A \subseteq C \cup D$ we have

$$(2.1) \quad \chi_{C,D} \setminus A = \chi_{C \setminus A, D \setminus A}.$$

For the second operation, assume \bar{r} is a round counter and we have a subset $S \subseteq \text{act } \bar{r}$. We let $\bar{r} \downarrow S$ denote the round counter defined by

$$(\bar{r} \downarrow S)(i) = \begin{cases} \bar{r}(i), & \text{if } i \notin S; \\ \bar{r}(i) - 1, & \text{if } i \in S. \end{cases}$$

We say that the round counter $\bar{r} \downarrow S$ is obtained from \bar{r} by the *execution* of S . Note that $\text{supp } (\bar{r} \downarrow S) = \text{supp } \bar{r}$, $\text{act } (\bar{r} \downarrow S) = \{i \in \text{act } \bar{r} \mid i \notin S, \text{ or } \bar{r}(i) \geq 2\}$, and $\text{pass } (\bar{r} \downarrow S) = \text{pass } \bar{r} \cup \{i \in S \mid \bar{r}(i) = 1\}$. However, in general we have $\chi(\bar{r}) \downarrow S \neq \chi(\bar{r} \downarrow S)$.

For an arbitrary round pointer \bar{r} and sets $S \subseteq \text{act } \bar{r}$, $A \subseteq \text{supp } \bar{r}$ we set

$$(2.2) \quad \bar{r}_{S,A} := (\bar{r} \downarrow S) \setminus A = (\bar{r} \setminus A) \downarrow (S \setminus A).$$

In the special case, when $A \cap S = \emptyset$, the identity (2.2) specializes to

$$(2.3) \quad \bar{r}_{S,A} := (\bar{r} \downarrow S) \setminus A = (\bar{r} \setminus A) \downarrow S.$$

When $A = \emptyset$, we shall frequently use the short-hand notation \bar{r}_S instead of $\bar{r}_{S,A}$, in other words, $\bar{r}_S = \bar{r} \downarrow S$. Again, for future reference, we note that for $S \subseteq C$, we have

$$(2.4) \quad \chi_{C,D} \downarrow S = \chi_{C \setminus S, D \cup S}.$$

Assume now we are given a round counter \bar{r} , and let $\varphi : \text{supp } \bar{r} \rightarrow [|\text{supp } \bar{r}| - 1]$ denote the unique order-preserving bijection. The round counter $c(\bar{r})$ is defined by

$$c(\bar{r})(i) := \begin{cases} \bar{r}(\varphi^{-1}(i)), & \text{for } 0 \leq i \leq |\text{supp } \bar{r}| - 1; \\ \perp, & \text{for } i \geq |\text{supp } \bar{r}|. \end{cases}$$

We call $c(\bar{r})$ the *canonical form* of \bar{r} . Note that $\text{supp } c(\bar{r}) = [|\text{supp } \bar{r}| - 1]$, $|\text{act } (c(\bar{r}))| = |\text{act } \bar{r}|$, and $|\text{pass } (c(\bar{r}))| = |\text{pass } (\bar{r})|$.

Let $\mathcal{S}_{\mathbb{Z}_+}$ denote the group of bijections $\pi : \mathbb{Z}_+ \rightarrow \mathbb{Z}_+$, such that $\pi(i) \neq \pi(i)$ for only finitely many i . This group acts on the set of all round counters, namely for $\pi \in \mathcal{S}_{\mathbb{Z}_+}$, and a round counter \bar{r} we set $\pi(\bar{r})(i) := \bar{r}(\pi(i))$.

2.3. Witness structures.

Definition 2.4. A **witness prestructure** is a sequence of pairs of finite subsets of \mathbb{Z}_+ , denoted $\sigma = ((W_0, G_0), \dots, (W_t, G_t))$, with $t \geq 0$, satisfying the following conditions:

- (P1) $W_i, G_i \subseteq W_0$, for all $i = 1, \dots, t$;
- (P2) $G_i \cap G_j = \emptyset$, for all $0 \leq i < j \leq t$;
- (P3) $G_i \cap W_j = \emptyset$, for all $0 \leq i \leq j \leq t$.

A witness prestructure is called **stable** if in addition the following condition is satisfied:

- (S) if $t \geq 1$, then $W_t \neq \emptyset$.

A **witness structure** is a witness prestructure satisfying the following strengthening of condition (S):

- (W) the subsets W_1, \dots, W_t are all nonempty.

W_0	W_1	W_2	\dots	W_t
G_0	G_1	G_2	\dots	G_t

FIGURE 2.1. Table presentation of a witness (pre)structure.

It is often useful to depict a witness prestructure in form of a table, see Figure 2.1. Note, that every witness prestructure with $t = 0$ is a witness structure. On the other hand, if $W_0 = \emptyset$, then conditions (P1) and (S) imply that $t = 0$. In this case, only the set G_0 carries any information, and we call this witness structure *empty*.

Definition 2.5. We define the following data associated to an arbitrary witness prestructure $\sigma = ((W_0, G_0), \dots, (W_t, G_t))$:

- the set $W_0 \cup G_0$ is called the **support** of σ and is denoted by $\text{supp } \sigma$;
- the **ghost set** of σ is the set $G(\sigma) := G_0 \cup \dots \cup G_t$;
- the **active set** of σ is the complement of the ghost set

$$A(\sigma) := \text{supp } (\sigma) \setminus G(\sigma) = W_0 \setminus (G_1 \cup \dots \cup G_t);$$

- the **dimension** of σ is

$$\dim \sigma := |A(\sigma)| - 1 = |W_0| - |G_1| - \dots - |G_t| - 1.$$

By definition, the dimension of a witness prestructure σ is between -1 and $|\text{supp } \sigma| - 1$. Let us analyze witness structures of special dimensions. To start with, if $\dim(\sigma) = -1$, then $A(\sigma) = \emptyset$. In particular, $W_0 = \emptyset$, hence $W_1 = \dots, W_t = \emptyset$ and $G_1 = \dots = G_t = \emptyset$. All witness structures of dimension -1 are empty, i.e., of the form $\sigma = ((\emptyset, G_0))$.

Furthermore, it is easy to characterize all witness structures σ of dimension 0. In this case, we have $|A(\sigma)| = 1$. We let $\sigma = ((W_0, G_0), \dots, (W_t, G_t))$ and let p denote the unique element of $A(\sigma)$. Then σ has dimension 0 if and only if

$$W_k \subseteq \{p\} \cup G_{k+1} \cup \dots \cup G_t, \text{ for all } k = 0, \dots, t.$$

In particular, we must of course have $W_t = \{p\}$. In such a case, we shall call p the *color* of the strict witness structure σ .

At the opposite extreme, a witness structure $\sigma = ((W_0, G_0), \dots, (W_t, G_t))$ has dimension $|\text{supp } \sigma| - 1$ if and only if $G_0 = \dots = G_t = \emptyset$. In such a case, we shall frequently use the short-hand notation $\sigma = (W_0, W_1, \dots, W_t)$.

For brevity of some formulas, we set $W_{-1} := W_0 \cup G_0 = \text{supp } \sigma$. Furthermore, we set $R_i := W_i \cup G_i$, for $i = 0, \dots, t$.

Definition 2.6. For a prestructure σ and an arbitrary $p \in \text{supp } \sigma$, we let $\text{Tr}(p, \sigma)$ denote the set $\{0 \leq i \leq t \mid p \in R_i\}$, which is called the **trace** of p . Furthermore, for all $p \in \text{supp } \sigma$, we set $\text{last}(p, \sigma) := \max\{i \mid p \in W_i\}$, and $M(p, \sigma) := |\text{Tr}(p)|$.

When the choice of σ is unambiguous, we shall simply write $\text{Tr}(p)$, $\text{last}(p)$, and $M(p)$. Note furthermore, that if $p \in A(\sigma)$, then $\text{Tr}(p) = \{0 \leq i \leq t \mid p \in W_i\}$, while if $p \in G(\sigma)$, then $\text{smax } \text{Tr}(p) = \{0 \leq i \leq t \mid p \in W_i\}$ and $p \in G_{\max \text{Tr}(p)}$.

To get a better grasp on the witness structures, as well as operations in them, the following alternative approach using traces is often of use.

Definition 2.7. A **witness prestructure** is a pair of finite subsets $A, G \subseteq \mathbb{Z}_+$ together with a family $(\text{Tr}(p))_{p \in A \cup G}$ of finite subsets of \mathbb{Z}_+ , satisfying the following two condition:

(T) $0 \in \text{Tr}(p)$, for all $p \in A \cup G$.

A witness prestructure is called **stable** if it satisfies the following additional condition:

(TS) if $A = \emptyset$, then $\text{Tr}(p) = \{0\}$, for all $p \in G$, else

$$\text{last}(p)_{p \in A} \geq \max(\text{Tr}(p))_{p \in G}.$$

Set $t := \text{last}(p)_{p \in A}$. The witness prestructure is called **witness structure** if the following strengthening of Condition (TS) is satisfied:

(TW) for all $1 \leq k \leq t$ either there exists $p \in A$ such that $k \in \text{Tr}(p)$, or there exists $p \in G$ such that $k \in \text{smax } \text{Tr}(p)$.

We shall call the form of the presentation of the witness prestructure described in Definition 2.7 its *trace form*.

Proposition 2.8. The Definitions 2.4 and 2.7 provide alternative descriptions of the same mathematical objects.

Proof. The translation between the two descriptions is as follows. First, assume $\sigma = ((W_0, G_0), \dots, (W_t, G_t))$ is a witness prestructure according to Definition 2.4. Set $A := A(\sigma)$, $G := G(\sigma)$, and for each $p \in A \cup G$, set $\text{Tr}(p)$ to be the trace of p as given by Definition 2.6.

Reversely, assume A , G , and $(\text{Tr}(p))_{p \in A \cup G}$. We set $t := \max(\text{Tr}(p))_{p \in A \cup G}$, and for all $0 \leq k \leq t$, we set

$$G_k := \{p \in G \mid k = \max \text{Tr}(p)\},$$

$$W_k := \{p \in A \cup G \mid k \in \text{Tr}(p)\} \setminus G_k.$$

We leave to the reader to verify that these translations are inverses of each other, that they preserve stability, and that they translate witness structures into witness structures. \square

2.4. Operations on witness prestructures.

2.4.1. Canonical form of a stable witness prestructure.

Any stable witness prestructure can be turned into a witness structure by means of the following operation.

Definition 2.9. Assume $\sigma = ((W_0, G_0), \dots, (W_t, G_t))$ is an arbitrary stable witness prestructure. Set $q := |\{1 \leq i \leq t \mid W_i \neq \emptyset\}|$. Pick $0 = i_0 < i_1 < \dots < i_q = t$, such that $\{i_1, \dots, i_q\} = \{1 \leq i \leq t \mid W_i \neq \emptyset\}$. We define the witness structure $C(\sigma) = ((W_0, G_0), (\widetilde{W}_1, \widetilde{G}_1), \dots, (\widetilde{W}_q, \widetilde{G}_q))$, which is called the **canonical form** of σ , by setting

$$(2.5) \quad \widetilde{W}_k := W_{i_k}, \quad \widetilde{G}_k := G_{i_{k-1}+1} \cup \dots \cup G_{i_k}, \text{ for all } k = 1, \dots, q,$$

The construction in Definition 2.9 is illustrated by Figure 2.2.

1, 2, 3, 4	\emptyset	2	\emptyset	1		1, 2, 3, 4	2	1
5	4	\emptyset	2	3	\longrightarrow	5	4	2, 3

FIGURE 2.2. A stable witness prestructure and its canonical form.

Proposition 2.10. Assume σ is an arbitrary stable witness prestructure.

- (a) The canonical form of σ is a well-defined witness structure.
- (b) We have $C(\sigma) = \sigma$ if and only if σ is a witness structure itself.
- (c) We have $\text{supp}(C(\sigma)) = \text{supp}(\sigma)$, $A(\sigma) = A(C(\sigma))$, $G(\sigma) = G(C(\sigma))$, and $\dim(\sigma) = \dim(C(\sigma))$.

Proof. Assume $\sigma = ((W_0, G_0), \dots, (W_t, G_t))$, q and i_1, \dots, i_q as in the Definition 2.9, and $c(\sigma) = ((W_0, G_0), (\widetilde{W}_1, \widetilde{G}_1), \dots, (\widetilde{W}_q, \widetilde{G}_q))$.

To prove (a) note first that all the set involved are finite subsets of \mathbb{Z}_+ . Conditions (P1) and (P2) for $C(\sigma)$ follow immediately from the corresponding conditions on σ . To see (P3), pick some $p \in \widetilde{G}_k$. Then there exists a unique j , such that $i_{k-1} < j \leq i_k$ and $p \in G_j$. Then $p \notin W_j \cup \dots \cup W_t$, but $W_j \cup \dots \cup W_t = W_{i_k} \cup \dots \cup W_{i_q}$, hence $p \notin \widetilde{W}_k \cup \dots \cup \widetilde{W}_q$. Finally, to see (W) note that $W_{i_k} \neq \emptyset$ for all $k = 1, \dots, q$, hence $\widetilde{W}_k \neq \emptyset$.

To prove (b) note that if σ is a witness structure, then $W_1, \dots, W_t \neq \emptyset$, hence $q = t$, $i_k = k$, for $k = 1, \dots, t$. It follows that $\widetilde{W}_k = W_k$, $\widetilde{G}_k = G_k$, for all $k = 1, \dots, t$. Reversely, assume $C(\sigma) = \sigma$, then $q = t$, hence $i_k = k$, for all $k = 1, \dots, t$, implying $W_1, \dots, W_t \neq \emptyset$.

To prove (c) note that the first pair of sets in σ and in $C(\sigma)$ is the same, hence $\text{supp}(C(\sigma)) = \text{supp}(\sigma)$. By (2.5) we have $\widetilde{W}_1 \cup \dots \cup \widetilde{W}_q = W_1 \cup \dots \cup W_t$, and $\widetilde{G}_1 \cup \dots \cup \widetilde{G}_q = G_1 \cup \dots \cup G_t$, hence $A(\sigma) = A(C(\sigma))$. The other two equalities follow. \square

2.4.2. Stabilization of witness prestructures.

Any witness prestructure can be made stable using the following operation.

Definition 2.11. Let $\sigma = ((W_0, G_0), \dots, (W_t, G_t))$ be a witness prestructure, and $S \subseteq A(\sigma)$. Set

$$q := \max\{0 \leq i \leq t \mid R_i \not\subseteq S \cup G(\sigma)\}.$$

The **stabilization** of σ is the witness prestructure $\text{st}_S(\sigma)$ whose trace form is $A(\sigma) \setminus S$, $G(\sigma) \cup S$, $(\text{Tr}(p)|_{\{0, \dots, q\}})_{p \in \text{supp } \sigma}$.

In the special case $S = \emptyset$ we can simply talk about the stabilization of a witness prestructure.

The following three properties provide an equivalent recursive definition of stabilization.

- (1) If $t = 0$, then $\text{st}_S(\sigma) = ((W_0 \setminus S, G_0 \cup S))$.
- (2) If $t \geq 1$ and $W_t \subseteq S$, then

$$\text{st}_S(\sigma) = \text{st}_{S \cup G_t}(((W_0, G_0), \dots, (W_{t-1}, G_{t-1}))).$$

- (3) If $t \geq 1$ and $W_t \not\subseteq S$, then the trace form of $\text{st}_S(\sigma)$ is $A(\sigma) \setminus S, G(\sigma) \cup S, (\text{Tr}(p))_{p \in A \cup G}$.

Assume now that $\text{st}_S(\sigma) = ((\widetilde{W}_0, \widetilde{G}_0), \dots, (\widetilde{W}_q, \widetilde{G}_q))$. By Definition 2.11 we have $\widetilde{W}_i \cup \widetilde{G}_i = R_i$, for all $0 \leq i \leq q$. Hence, for some sets J_0, \dots, J_q we have

$$(2.6) \quad \text{st}_S(\sigma) = ((W_0 \setminus J_0, G_0 \cup J_0), \dots, (W_q \setminus J_q, G_q \cup J_q)).$$

We shall refer to (2.6) as the table form of $\text{st}_S(\sigma)$. The sets J_i are explicitly described by the following formula:

$$J_i := (W_i \setminus (W_{i+1} \cup \dots \cup W_q)) \cap (S \cup G(\sigma)).$$

1, 2, 3, 4, 5	1	3, 4, 5	2, 3	1	1	\emptyset	\rightarrow	1, 3, 4, 5	\emptyset	4, 5
\emptyset	\emptyset	\emptyset	\emptyset	3	2	1		2	1	3

FIGURE 2.3. Stabilizing a witness prestructure for $S = \{\emptyset\}$.

Proposition 2.12. *Assume as before that we are given a strict witness structure σ , and $S \subsetneq A(\sigma)$. The witness prestructure $\text{st}_S(\sigma)$ is well-defined and stable. It satisfies the following properties:*

- $\text{supp}(\text{st}_S(\sigma)) = \text{supp } \sigma$;
- $G(\text{st}_S(\sigma)) = G(\sigma) \cup S$;
- $A(\text{st}_S(\sigma)) = A(\sigma) \setminus S$;
- $\dim \text{st}_S(\sigma) = \dim \sigma - |S|$.

Proof. Straightforward verification. □

The following property if the stabilization will be very useful later on.

Proposition 2.13. *Assume σ is a witness prestructure, and $S, T \subseteq A(\sigma)$, such that $S \cap T = \emptyset$. Then we have*

$$(2.7) \quad \text{st}_T(\text{st}_S(\sigma)) = \text{st}_{S \cup T}(\sigma).$$

Proof. Let $\sigma' = \text{st}_T(\text{st}_S(\sigma))$ and $\sigma'' = \text{st}_{S \cup T}(\sigma)$. To show that $\sigma' = \sigma''$ we compare their trace forms. To start with, by Definition 2.11 we have $\text{supp } \sigma' = \text{supp } \sigma$ and $\text{supp } \sigma'' = \text{supp } \sigma$. Furthermore, $A(\sigma'') = A(\sigma) \setminus (S \cup T)$, and $A(\sigma') = A(\text{st}_S(\sigma)) \setminus T = (A(\sigma) \setminus S) \setminus T$, hence $A(\sigma') = A(\sigma'')$ and $G(\sigma) = G(\sigma'')$.

It remains to show that the traces of the elements from $\text{supp } \sigma$ are truncated at the same index in σ' and in σ'' . For σ'' the traces are truncated at $q = \max\{0 \leq i \leq t \mid R_i \not\subseteq S \cup T \cup G(\sigma)\}$. On the other hand, to obtain $\text{st}_S(\sigma)$ we truncate at $q' = \max\{0 \leq i \leq t \mid R_i \not\subseteq S \cup G(\sigma)\}$. Assume $\text{st}_S(\sigma) = ((\widetilde{W}_0, \widetilde{G}_0), \dots, (\widetilde{W}_{q'}, \widetilde{G}_{q'}))$. We have $\widetilde{W}_i \cup \widetilde{G}_i = R_i$, for all $0 \leq i \leq q'$. To obtain σ' from $\text{st}_S(\sigma)$ we now truncate

the traces in $\text{st}_S(\sigma)$ at $q'' = \max\{0 \leq j \leq q' \mid R_j \not\subseteq T \cup G(\text{st}_S(\sigma))\}$. Since $q' \geq q$, and $G(\text{st}_S(\sigma)) = G(\sigma) \cup S$, we obtain $q = q''$. It follows that $\sigma' = \sigma''$. \square

2.4.3. Ghosting operation on the witness structures.

We are now ready to define the main operation on witness structures.

Definition 2.14. We define $\Gamma_S(\sigma) := C(\text{st}_S(\sigma))$. We say that $\Gamma_S(\sigma)$ is obtained from σ by **ghosting** S .

The ghosting operation is illustrated on Figure 2.4. When $S = \{p\}$, we shall simply write $\Gamma_p(\sigma)$.

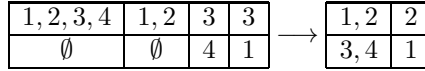


FIGURE 2.4. Ghosting a witness structure for $S = \{3\}$.

Clearly, we have $\Gamma_\emptyset(\sigma) = \sigma$. As the next step, if $S = \{p\}$, i.e., we are ghosting a single element, the situation is not quite straightforward, though several special cases can be formulated simpler.

Let $l := \text{last}(p)$. If $|W_l| \geq 2$, then the situation is much simpler indeed. In this case $J_i = \emptyset$, for all $i \neq l$, while $J_l = \{p\}$. Accordingly, we get

$$\Gamma_p(\sigma) = ((W_0, G_0), \dots, (W_{l-1}, G_{l-1}), (W_l \setminus \{p\}, G_l \cup \{p\}), (W_{l+1}, G_{l+1}), \dots, (W_t, G_t)).$$

The situation is slightly more complex if $|W_l| = 1$, i.e., $W_l = \{p\}$. Assume that $l \leq t-1$. Then, we still have $J_i = \emptyset$, for all $i \neq l$, and $J_l = \{p\}$. The difference now is that

$$\tilde{G} = ((W_0, G_0), \dots, (W_{l-1}, G_{l-1}), (\emptyset, G_l \cup \{p\}), (W_{l+1}, G_{l+1}), \dots, (W_t, G_t))$$

is now only a witness prestructure, so in this case we get

$$\Gamma_p(\sigma) = ((W_0, G_0), \dots, (W_{l-1}, G_{l-1}), (W_{l+1}, G_l \cup \{p\} \cup G_{l+1}), \dots, (W_t, G_t)).$$

Once $l = t$, i.e., $W_t = \{p\}$, we will need the full strength of the Definition 2.14.

The situation is similar if $|S| \geq 2$. For each element $s \in S$ we set $l(s) := \text{last}(s)$. As long as each $W_{l(s)}$ contains elements outside of S , all that happens is that each element $s \in S$ gets moved from $W_{l(s)}$ to $G_{l(s)}$. Once this is not true, a more complex construction is needed.

Proposition 2.15. Assume as before that we are given a strict witness structure $\sigma = ((W_0, G_0), \dots, (W_t, G_t))$, and $S \subsetneq A(\sigma)$. The construction in Definition 2.14 is well-defined, and yields a witness structure $\Gamma_S(\sigma)$, satisfying the following properties:

- $\text{supp}(\Gamma_S(\sigma)) = \text{supp } \sigma$;
- $G(\Gamma_S(\sigma)) = G(\sigma) \cup S$;
- $A(\Gamma_S(\sigma)) = A(\sigma) \setminus S$;
- $\dim \Gamma_S(\sigma) = \dim \sigma - |S|$.

Proof. All equalities follow from the Propositions 2.10 and 2.12. \square

Remark 2.16. For future reference we make the following observation. Let $\sigma = ((W_0, G_0), \dots, (W_t, G_t))$, and $p, q \in \text{supp } \sigma$. We always have $M(q, \Gamma_p(\sigma)) = M(q, \sigma)$, except for one single case: namely, when $p = q$ and $W_t = \{p\}$, we have the strict inequality $M(p, \Gamma_p(\sigma)) < M(p, \sigma)$.

Lemma 2.17. Assume σ is a stable prestructure, and $S \subseteq A(\sigma)$, then we have $C(\text{st}_S(\sigma)) = C(\text{st}_S(C(\sigma)))$, or expressed functorially $C \circ \text{st}_S \circ C = C \circ \text{st}_S$.

Proof. Assume $\sigma = ((W_0, G_0), (W_1, G_1), \dots, (W_t, G_t))$, and set $\tilde{S} := S \cup G(\sigma)$. For appropriately chosen q and $0 = i_0 < i_1 < \dots < i_q$ we have

$$C(\sigma) = ((W_{i_0}, \tilde{G}_{i_0}), (W_{i_1}, \tilde{G}_{i_1}), \dots, (W_{i_q}, \tilde{G}_{i_q})),$$

where $\tilde{G}_{i_k} = \bigcup_{\alpha=i_{k-1}+1}^{i_k} G_\alpha$, for $k = 0, \dots, q$.

Set $r := \max\{1 \leq k \leq q \mid W_{i_k} \not\subseteq \tilde{S}\}$, and $J_k = (W_{i_k} \setminus \bigcup_{\alpha=k+1}^q W_{i_\alpha}) \cap \tilde{S}$, for $0 \leq k \leq r$. Then $i_r := \max\{1 \leq j \leq t \mid W_j \not\subseteq \tilde{S}\}$ and $J_k = (W_{i_k} \setminus \bigcup_{\alpha=i_k+1}^t W_\alpha) \cap \tilde{S}$, for $0 \leq k \leq r$, since $W_j = \emptyset$ whenever $j \notin \{i_0, i_1, \dots, i_q\}$. It follows that

$$\text{st}_S(C(\sigma)) = ((W_{i_0} \setminus J_0, \tilde{G}_{i_0} \cup J_0), (W_{i_1} \setminus J_1, \tilde{G}_{i_1} \cup J_1), \dots, (W_{i_r} \setminus J_r, \tilde{G}_{i_r} \cup J_r)).$$

On the other hand, we have $\text{st}_S(\sigma) = ((W'_0, G'_0), (W'_1, G'_1), \dots, (W'_{i_r}, G'_{i_r}))$, where

$$(2.8) \quad W'_j = \begin{cases} W_{i_k} \setminus J_k, & \text{if } j = i_k, \\ \emptyset, & \text{otherwise;} \end{cases} \quad G'_j = \begin{cases} G_{i_k} \cup J_k, & \text{if } j = i_k, \\ G_j, & \text{otherwise.} \end{cases}$$

Set $d := |\{1 \leq k \leq r \mid W_{i_k} \setminus J_k \neq \emptyset\}|$, then (2.8) implies that we also have $d = |\{1 \leq k \leq i_r \mid W'_k \neq \emptyset\}|$. This means that $C(\text{st}_S(\sigma))$ and $C(\text{st}_S(C(\sigma)))$ have the same length.

For the appropriate choice of $0 = a(0) < a(1) < \dots < a(d) = r$ we have

$$\{a(1), \dots, a(d)\} = \{1 \leq k \leq r \mid W_{i_k} \setminus J_k \neq \emptyset\}.$$

Assume $C(\text{st}_S(C(\sigma))) = ((V_0, H_0), \dots, (V_d, H_d))$, then we have $V_k = W_{i_{a(k)}} \setminus J_{a(k)}$,

$$(2.9) \quad H_k = \bigcup_{\alpha=a(k-1)+1}^{a(k)} (\tilde{G}_{i_\alpha} \cup J_\alpha) = \bigcup_{\alpha=a(k-1)+1}^{a(k)} \tilde{G}_{i_\alpha} \cup \bigcup_{\alpha=a(k-1)+1}^{a(k)} J_\alpha,$$

for $0 \leq k \leq d$.

Assume now that $C(\text{st}_S(\sigma)) = ((V'_0, H'_0), \dots, (V'_d, H'_d))$. Note that

$$\{i_{a(1)}, \dots, i_{a(d)}\} = \{1 \leq k \leq i_r \mid W'_k \neq \emptyset\},$$

hence, for $0 \leq k \leq d$, we get $V'_k = W'_{i_{a(k)}} = W_{i_{a(k)}} \setminus J_{a(k)}$, and

$$H'_k = \bigcup_{\alpha=i_{a(k-1)}+1}^{i_{a(k)}} G'_\alpha = \bigcup_{\alpha=i_{a(k-1)}+1}^{i_{a(k)}} G_\alpha \cup \bigcup_{\alpha=a(k-1)+1}^{a(k)} J_\alpha,$$

where the last equality is a consequence of (2.8). Combining the identity

$$\bigcup_{\alpha=a(k-1)+1}^{a(k)} \tilde{G}_{i_\alpha} = \bigcup_{\alpha=a(k-1)+1}^{a(k)} \bigcup_{\beta=i_{\alpha-1}+1}^{i_\alpha} G_\beta = \bigcup_{\beta=i_{a(k-1)}+1}^{i_{a(k)}} G_\beta$$

with (2.9), we see that $H_k = H'_k$, for all $0 \leq k \leq d$. \square

Proposition 2.18. *Assume σ is a witness structure, and $S, T \subseteq A(\sigma)$, such that $S \cap T = \emptyset$. Then we have $\Gamma_T(\Gamma_S(\sigma)) = \Gamma_{S \cup T}(\sigma)$, expressed functorially we have $\Gamma_T \circ \Gamma_S = \Gamma_{S \cup T}$.*

Proof. We have

$$\Gamma_T \circ \Gamma_S = C \circ \text{st}_T \circ C \circ \text{st}_S = C \circ \text{st}_T \circ \text{st}_S = C \circ \text{st}_{S \cup T} = \Gamma_{S \cup T},$$

where the first and the fourth equalities follow from Definition 2.14, the second equality follows from Lemma 2.17, and the third equality follows from Proposition 2.13. \square

3. IMMEDIATE SNAPSHOT COMPLEXES

3.1. Combinatorial definition.

We now define our main objects of study.

Definition 3.1. *Assume \bar{r} is a round counter. We define a simplicial complex $P(\bar{r})$ as follows:*

- *the simplices of $P(\bar{r})$ are indexed by witness structures σ satisfying the following properties:*
 - (1) $\text{supp } \sigma = \text{supp } \bar{r}$;
 - (2) *for all $p \in A(\sigma)$, we have $|M(p, \sigma)| = r(p) + 1$;*
 - (3) *for all $p \in G(\sigma)$, we have $|M(p, \sigma)| \leq r(p) + 1$.*
- *the dimension of the simplex indexed by σ is $\dim \sigma$, its vertices are $\Gamma_{A(\sigma) \setminus \{a\}}(\sigma)$, where a ranges through the set $A(\sigma)$.*

*The complex $P(\bar{r})$ is called the **immediate snapshot complex** associated to the round counter \bar{r} .*

Assume \bar{r} is a round counter, such that $\bar{r}(i) = \perp$ for all $i \geq n + 1$. In line with our short-hand notation for the round counters, and in addition skipping a pair of brackets, we shall use an alternative notation $P(\bar{r}(0), \dots, \bar{r}(n))$ instead of $P(\bar{r})$. For every $\sigma \in P(\bar{r})$ we shall write $V(\sigma)$ to denote the set of vertices of σ . We shall also for brevity often identify witness structures with the simplices which they are indexing, e.g., saying that $\sigma \supseteq \tau$ to indicate that the simplex indexed by σ contains the simplex indexed by τ .

The next proposition checks that the Definition 3.1 yields a well-defined simplicial complex, and shows that the ghosting operation provides the right combinatorial language to describe boundaries in $P(\bar{r})$.

Proposition 3.2. *Assume \bar{r} is the round counter.*

- (1) *The associated immediate snapshot complex $P(\bar{r})$ is a well-defined simplicial complex.*
- (2) *Assume σ and τ are simplices of $P(\bar{r})$. Then $\tau \subseteq \sigma$ if and only if there exists $S \subseteq A(\sigma)$, such that $\tau = \Gamma_S(\sigma)$.*

Proof. We start by showing (1). We have already observed that the only witness structure of dimension -1 is the empty one. Since $\text{supp } \sigma = \text{supp } \bar{r}$, the complex $P(\bar{r})$ has exactly one simplex of dimension -1 , namely $(\emptyset, \text{supp } \bar{r})$.

Assume now that the witness structure σ indexes a simplex of $P(\bar{r})$. Set $d := \dim \sigma$, implying that $A(\sigma) = \{p_0, \dots, p_d\}$ for $p_0 < \dots < p_d$, $p_i \in \mathbb{Z}_+$. For $0 \leq i \leq d$,

we set $v_i := \Gamma_{A(\sigma) \setminus \{p\}}(\sigma)$. We see that the d -dimensional simplex σ has $d+1$ vertices, which are all distinct, since $A(v_i) = p_i$, for $0 \leq i \leq d$. Furthermore, it follows from the Reconstruction Lemma 3.3 that any two simplices with the same set of vertices are equal, implying that the simplicial complex $P(\bar{r})$ is well-defined.

Let us now show (2). To start with, assume $\tau = \Gamma_S(\sigma)$, for some $S \subseteq A(\sigma)$. By Proposition 2.15 we have $A(\tau) = A(\sigma) \setminus S$. It follows from Proposition 2.18 that for every $p \in A(\tau)$ we have

$$\Gamma_{A(\tau) \setminus \{p\}}(\tau) = \Gamma_{A(\tau) \setminus \{p\}}(\Gamma_S(\sigma)) = \Gamma_{A(\tau) \cup S \setminus \{p\}}(\sigma) = \Gamma_{A(\sigma) \setminus \{p\}}(\sigma),$$

hence the set of vertices of τ is a subset of the set of vertices of σ .

Reversely, assume $V(\tau) \subseteq V(\sigma)$. The same computation as above shows, that $V(\Gamma_{\text{supp } \sigma \setminus \text{supp } \tau}(\sigma)) = \text{supp } \tau$, i.e., τ and $\Gamma_{\text{supp } \sigma \setminus \text{supp } \tau}(\sigma)$ have the same set of vertices. It follows from the Reconstruction Lemma 3.3 that $\tau = \Gamma_{\text{supp } \sigma \setminus \text{supp } \tau}(\sigma)$, and so (2) is proved. \square

3.2. The Reconstruction Lemma.

From the point of view of distributed computing, the vertices of $P(\bar{r})$ should be thought of as *local views* of specific processors. In this intuitive picture, the next Reconstruction Lemma 3.3 says that any set of local views corresponds to at most one global view.

Lemma 3.3. (Reconstruction Lemma).

Assume σ and τ are witness structures, such that the corresponding d -simplices of $P(\bar{r})$ have the same set of vertices, then we must have $\sigma = \tau$.

Proof. Assume the statement of lemma is not satisfied, and pick a pair of d -dimensional simplices $\sigma \neq \tau$, such that $V(\sigma) = V(\tau)$, and d is minimal possible. Obviously, we must have $d \geq 1$.

To start with, the set of vertices defines the support set, so $\text{supp } \sigma = \text{supp } \tau = \Sigma$. Let $p, q \in \Sigma$, then it is easy to check that $M(q, \sigma) = M(q, \Gamma_p(\sigma))$, and $M(p, \sigma) \leq M(p, \Gamma_p(\sigma))$. This means that the Σ -tuples $(M(p, \sigma))_{p \in \Sigma}$ and $(M(p, \tau))_{p \in \Sigma}$ are equal.

Let $\sigma = ((W_0, G_0), \dots, (W_t, G_t))$. Assume there exists $0 \leq k \leq t$, such that $|W_k \cap \Sigma| \geq 2$. Pick $p, q \in W_k \cap \Sigma$, $p \neq q$, then

$$\Gamma_p(\sigma) = \begin{array}{|c|c|c|c|c|c|c|c|} \hline W_0 & \dots & W_{k-1} & W_k \setminus \{p\} & W_{k+1} & \dots & W_t \\ \hline G_0 & \dots & G_{k-1} & G_k \cup \{p\} & G_{k+1} & \dots & G_t \\ \hline \end{array},$$

since $\Gamma_p(\sigma) = \Gamma_p(\tau)$, but $\sigma \neq \tau$, we get

$$(3.1) \quad \tau = \begin{array}{|c|c|c|c|c|c|c|c|} \hline W_0 & \dots & W_{k-1} & p & W_k \setminus \{p\} & W_{k+1} & \dots & W_t \\ \hline G_0 & \dots & G_{k-1} & A_p & B_p & G_{k+1} & \dots & G_t \\ \hline \end{array},$$

for some A_p, B_p such that $A_p \cup B_p = G_k$. Repeating the same argument with q instead of p we get

$$(3.2) \quad \tau = \begin{array}{|c|c|c|c|c|c|c|c|} \hline W_0 & \dots & W_{k-1} & q & W_k \setminus \{q\} & W_{k+1} & \dots & W_t \\ \hline G_0 & \dots & G_{k-1} & A_q & B_q & G_{k+1} & \dots & G_t \\ \hline \end{array},$$

for some A_q, B_q such that $A_q \cup B_q = G_k$. The equations (3.1) and (3.2) contradict each other. It is thus safe to assume that $|W_k \cap \Sigma| \leq 1$, and that the same is true for τ . An alternative way to phrase the same condition is to say that $\text{last}(p, \sigma) \neq \text{last}(q, \sigma)$, and $\text{last}(p, \tau) \neq \text{last}(q, \tau)$, for all $p, q \in \Sigma$.

Set $F := \{p \in \Sigma \mid M(p, \sigma) = M(p, \Gamma_p(\sigma))\}$. Note that $F = \{p \in \Sigma \mid M(p, \tau) = M(p, \Gamma_p(\tau))\}$. Using Remark 2.16, the previous observation $M(p, \sigma) \leq M(p, \Gamma_p(\sigma))$ can be strengthened as follows: we know that $F = \Sigma \setminus \{l\}$, for some $l \in \Sigma$. Specifically, $W_t = \{l\}$, and the last pair of sets in τ is also $(\{l\}, H)$, for some $H \subseteq G(\tau)$.

Pick $p \in F$ such that $\text{last}(p) = \max_{q \in F} \text{last}(q)$. Assume

$$\Gamma_p(\sigma) = \begin{array}{|c|c|c|c|c|c|c|} \hline W_0 & \dots & W_{k-1} & W_k & W_{k+1} & \dots & W_t \\ \hline G_0 & \dots & G_{k-1} & G_k \cup \{p\} & G_{k+1} & \dots & G_t \\ \hline \end{array}.$$

We observe, that p was chosen so that $(W_k \cup \dots \cup W_t) \cap F = \emptyset$. We can easily describe the set Λ of all d -simplices γ , for which $p \in \text{supp } \gamma$ and $\Gamma_p(\gamma) = \Gamma_p(\sigma)$. Set

$$\gamma^p := \begin{array}{|c|c|c|c|c|c|c|} \hline W_0 & \dots & W_{k-1} & W_k \cup \{p\} & W_{k+1} & \dots & W_t \\ \hline G_0 & \dots & G_{k-1} & G_k & G_{k+1} & \dots & G_t \\ \hline \end{array},$$

and

$$\gamma_{A,B} := \begin{array}{|c|c|c|c|c|c|c|} \hline W_0 & \dots & W_{k-1} & p & W_k & W_{k+1} & \dots & W_t \\ \hline G_0 & \dots & G_{k-1} & A & B & G_{k+1} & \dots & G_t \\ \hline \end{array},$$

where $A \cup B = G_k$. Then $\Lambda = \{\gamma_{A,B} \mid A \cup B = G_k\} \cup \{\gamma^p\}$. Clearly, $\sigma, \tau \in \Lambda$. We shall show that $\Gamma_l(\sigma) \neq \Gamma_l(\tau)$.

Assume $A \cup B = G_k$, and pick $\alpha \in W_k$. Then

$$M(\alpha, \Gamma_l(\gamma^p)) = \sum_{i=0}^{k-1} \chi(\alpha, R_i) + 1 \neq \sum_{i=0}^{k-1} \chi(\alpha, R_i) = M(\alpha, \Gamma_l(\gamma_{A,B})),$$

hence $\Gamma_l(\gamma^p) \neq \Gamma_l(\gamma_{A,B})$.

Assume now we have further sets A' and B' , such that $A' \cup B' = G_k$, $A \neq A'$. Without loss of generality, we can assume that $A \not\subseteq A'$. Pick now $\alpha \in A \setminus A'$. Then

$$M(\alpha, \Gamma_l(\gamma_{A,B})) = \sum_{i=0}^{k-1} \chi(\alpha, R_i) + 1 \neq \sum_{i=0}^{k-1} \chi(\alpha, R_i) = M(\alpha, \Gamma_l(\gamma_{A',B'})),$$

hence $\Gamma_l(\gamma_{A,B}) \neq \Gamma_l(\gamma_{A',B'})$.

We have thus proved that $\Gamma_l(\sigma) \neq \Gamma_l(\tau)$, contradicting the choice of σ and τ . \square

4. SOME OBSERVATIONS ON IMMEDIATE SNAPSHOT COMPLEXES

4.1. Elementary properties and examples.

We start by listing a few simple but useful properties of the immediate snapshot complexes $P(\bar{r})$.

First, for an arbitrary point counter \bar{r} , we have

$$(4.1) \quad P(\bar{r}) \simeq P(c(\bar{r})),$$

where \simeq denotes an isomorphism of simplicial complexes. Specifically, this isomorphism is given by the map

$$\varphi : ((W_0, G_0), \dots, (W_t, G_t)) \mapsto ((\varphi(W_0), \varphi(G_0)), \dots, (\varphi(W_t), \varphi(G_t))),$$

where φ is the unique order-preserving bijection $\varphi : \text{supp } \bar{r} \rightarrow [\text{supp } \bar{r}] - 1$. In particular, if round counters \bar{r} and \bar{q} have the same canonical form, then the corresponding immediate snapshot complexes are isomorphic. In other words, the \perp entries do not matter for the simplicial structure.

In a similar vein, for any round counter \bar{r} , and any permutation $\pi \in \mathcal{S}_{\mathbb{Z}_+}$, the simplicial complex $P(\pi(\bar{r}))$ is isomorphic to the simplicial complex $P(\bar{r})$. The isomorphism is given by the map

$$\varphi : ((W_0, G_0), \dots, (W_t, G_t)) \mapsto ((\pi(W_0), \pi(G_0)), \dots, (\pi(W_t), \pi(G_t))).$$

Let us now look at special round counters. If $\bar{r} = (r)$, then the simplicial complex $P(\bar{r})$ is just a point indexed by the witness structure $\underbrace{((0, \emptyset), \dots, (0, \emptyset))}_{r+1}$. Recall, that the empty simplex of $P(r)$ is indexed by the witness structure $((\emptyset, 0))$.

If $\bar{r} = \underbrace{(0, \dots, 0)}_{n+1}$, then $P(\bar{r})$ is isomorphic to the n -simplex Δ^n . The simplices of $P(\bar{r})$ are indexed by all $((A, B))$ such that $A \cap B = \emptyset$ and $A \cup B = [n]$. The simplicial isomorphism between $P(\bar{r})$ and Δ^n is given by $((A, B)) \mapsto A$. More generally, if \bar{r} is a round counter such that $r(i) \in \{\perp, 0\}$, for all $i \in \mathbb{Z}_+$, the simplicial complex $P(\bar{r})$ is isomorphic with $\Delta^{\text{supp } \bar{r}}$.

Assume now $\bar{r} = (r(0), \dots, r(n))$ and $\bar{r}(n) = 0$. Consider related round counter $\bar{q} := (r(0), \dots, r(n-1))$. Consider a cone over $P(\bar{q})$, which we denote $P(\bar{q}) * \{a\}$, where a is the apex of the cone. Then we have

$$(4.2) \quad P(\bar{r}) \simeq P(\bar{q}) * \{a\},$$

with the isomorphism given by

$$((W_0, G_0), \dots, (W_t, G_t)) \mapsto \begin{cases} ((W_0 \setminus \{n\}, G_0), \dots, (W_t, G_t)) * \{a\}, & \text{if } n \in W_0; \\ ((W_0, G_0 \setminus \{n\}), \dots, (W_t, G_t)), & \text{if } n \in G_0. \end{cases}$$

This observation can be iterated, so that all 0 entries in \bar{r} are replaced with the iterated cone construction.

The properties above can be summarized on the intuitive level as telling us that if we are interested in understanding the simplicial structure of the complex $P(\bar{r})$, we may ignore the entries \perp and 0, and permute the remaining entries as we see fit.

4.2. The purity of the immediate snapshot complexes.

Assume $\sigma = ((W_0, G_0), \dots, (W_t, G_t))$ is a witness structure which indexes a simplex of $P(\bar{r})$. Clearly, we have $|A(\sigma)| \leq |\text{supp } \bar{r}|$, hence $\dim \sigma \leq |\text{supp } \bar{r}| - 1$. It turns out that every simplex can be extended to the one having dimension $|\text{supp } \bar{r}| - 1$, implying that immediate snapshot complexes are always pure.

Proposition 4.1. *The simplicial complex $P(\bar{r})$ is pure of dimension $|\text{supp } \bar{r}| - 1$.*

Proof. Assume $\sigma = ((W_0, G_0), \dots, (W_t, G_t))$ is a witness structure which indexes a simplex of $P(\bar{r})$. For each $p \in G(\sigma)$ we set $m(p) := r(p) + 1 - |M(p, \sigma)|$. By construction, we have $m(p) \geq 0$. Set furthermore $q := \max_{p \in G(\sigma)} m(p)$,

$$V_i = \{p \in G(\sigma) \mid m(p) \geq i\}, \text{ for } i = 1, \dots, q,$$

and

$$\tilde{\sigma} := (W_0 \cup G_0, W_1 \cup G_1, \dots, W_t \cup G_t, V_1, \dots, V_q).$$

We see that $\tilde{\sigma}$ is a witness structure: the condition (P1) says that $V_i \subseteq W_0 \cup G_0$, the conditions (P2) and (P3) are immediate, and condition (W) says that $V_i \neq \emptyset$. Furthermore, we have $\text{supp } \tilde{\sigma} = \text{supp } \sigma$, $G(\tilde{\sigma}) = \emptyset$, and $A(\tilde{\sigma}) = \text{supp } \sigma = A(\sigma) \cup G(\sigma)$. For all $\sigma \in A(\sigma)$ we have $|M(p, \tilde{\sigma})| = |M(p, \sigma)| = r(p) + 1$, while for

all $\sigma \in G(\sigma)$ we have $|M(p, \tilde{\sigma})| = |M(p, \sigma)| + m(p) = r(p) + 1$. We conclude that $\tilde{\sigma}$ indexes simplex of $P(\bar{r})$. Clearly, $\dim \tilde{\sigma} = |\text{supp } \sigma| - 1$. Finally, we have $\Gamma(\tilde{\sigma}, G(\sigma)) = \sigma$, so $\tilde{\sigma} \subseteq \sigma$ and hence $P(\bar{r})$ is pure of dimension $|\text{supp } \bar{r}| - 1$. \square

4.3. Immediate snapshot complexes of dimension 1.

It follows from the above, that $\dim P(\bar{r}) = 0$ if and only if $|\text{supp } \bar{r}| = 1$, meaning that $P(\bar{r})$ is a point. Assume now $\dim P(\bar{r}) = 1$. In this case, we have $|\text{supp } \bar{r}| = 2$. By (4.1), up to the simplicial isomorphism, we can assume that $\bar{r} = (m, n)$, $m, n \geq 0$.

Proposition 4.2. *For any integers $m, n \geq 0$, the simplicial complex $P(m, n)$ is a subdivided interval.*

Proof. The simplicial complex $P(m, n)$ is a pure 1-dimensional complex. Hence, it is enough to directly verify that all vertices have valency 2, except for the vertices $((0, 1), (0, \emptyset), \dots, (0, \emptyset))$ and $((1, 0), (1, \emptyset), \dots, (1, 0))$, which have valency 1. \square

Let $f(m, n)$ denote the number of 1-simplices in $P(m, n)$. This number completely describes the complex $P(m, n)$.

Proposition 4.3. *The numbers $f(m, n)$ satisfy the recursive relation*

$$(4.3) \quad f(m, n) = f(m, n-1) + f(m-1, n) + f(m-1, n-1), \quad \forall m, n \geq 1,$$

with the boundary conditions $f(m, 0) = f(0, m) = 1$. The corresponding generating function

$$F(x, y) = \sum_{m, n=0}^{\infty} f(m, n) x^m y^n$$

is given by the following explicit formula:

$$(4.4) \quad F(x, y) = \frac{1}{1 - x - y - xy}.$$

Proof. Multiply (4.3) with $x^m y^n$ and sum over all m, n . \square

4.4. Number of simplices of maximal dimension in an immediate snapshot complex.

For arbitrary nonnegative integers m_0, \dots, m_n we let $f(m_0, \dots, m_n)$ denote the number of top-dimensional simplices in $P(m_0, \dots, m_n)$. Note that $f(m_0, \dots, m_n) = f(m_{\pi(0)}, \dots, m_{\pi(n)})$ for any $\pi \in \mathcal{S}_{[n]}$.

Proposition 4.4. *We have $f(m_0, \dots, m_{n-1}, 0) = f(m_0, \dots, m_{n-1})$. Furthermore, if $m_0, \dots, m_n \geq 1$, we have*

$$(4.5) \quad f(m_0, \dots, m_n) = \sum_{\emptyset \neq S \subseteq [n]} f(m_0^S, \dots, m_n^S),$$

where

$$m_k^S = \begin{cases} m_k - 1, & \text{if } k \in S; \\ m_k, & \text{if } k \notin S. \end{cases}$$

Proof. Immediate consequence of the canonical decomposition of $P(m_0, \dots, m_n)$. \square

4.5. Standard chromatic subdivision as immediate snapshot complex.

The *standard chromatic subdivision* of an n -simplex, denoted $\chi(\Delta^n)$, is a prominent and much studied structure in distributed computing. We refer to [HKR, HS] for distributed computing background, and to [Ko12, Ko13] for the analysis of its simplicial structure.

In particular the following combinatorial description of $\chi(\Delta^n)$ has been given in [Ko13]. The top dimensional simplices of $\chi(\Delta^n)$ are indexed by ordered tuples of disjoint sets (B_1, \dots, B_t) such that $B_1 \cup \dots \cup B_t = [n]$. The lower dimensional simplices of $\chi(\Delta^n)$ are indexed by pairs of tuples of non-empty sets $((B_1, \dots, B_t)(C_1, \dots, C_t))$, such that B_i 's are disjoint subsets of $[n]$, and $C_i \subseteq B_i$ for all i . For brevity, set $P_n := P(\underbrace{1, \dots, 1}_{n+1})$.

Proposition 4.5. *The immediate snapshot complex P_n and the standard chromatic subdivision of an n -simplex are isomorphic as simplicial complexes. Explicitly, the isomorphism can be given by*

$$(4.6) \quad ((B_1, \dots, B_t)(C_1, \dots, C_t)) \mapsto \begin{array}{|c|c|c|c|c|} \hline W_0 & C_1 & C_2 & \dots & C_t \\ \hline [n] \setminus W_0 & B_1 \setminus C_1 & B_2 \setminus C_2 & \dots & B_t \setminus C_t \\ \hline \end{array},$$

where $W_0 = B_1 \cup \dots \cup B_t$.

Proof. □

Note that (4.6) yields a direct description of the simplicial structure of P_n , namely the simplices of P_n are indexed by all witness structures $\sigma = ((W_0, G_0), \dots, (W_t, G_t))$ satisfying the following conditions:

- (1) $W_0 \cup G_0 = [n]$;
- (2) $W_0 = W_1 \cup \dots \cup W_t \cup G_1 \cup \dots \cup G_t$;
- (3) the sets $W_1, \dots, W_t, G_1, \dots, G_t$ are disjoint.

5. A CANONICAL DECOMPOSITION OF THE IMMEDIATE SNAPSHOT COMPLEXES

5.1. Definition and examples.

Definition 5.1. *Assume \bar{r} is a round counter.*

- For every subset $S \subseteq \text{act } \bar{r}$, let Z_S denote the set of all simplices $\sigma = ((W_0, G_0), \dots, (W_t, G_t))$, such that $S \subseteq G_1$.
- For every pair of subsets $A \subseteq S \subseteq \text{act } \bar{r}$, let $Y_{S,A}$ denote the set of all simplices $\sigma = ((W_0, G_0), \dots, (W_t, G_t))$, such that $R_1 = S$ and $A \subseteq G_1$. Furthermore, set $X_{S,A} := Y_{S,A} \cup Z_S$.

We shall also use the following short-hand notation: $X_S := X_{S,\emptyset}$. On the other extreme, clearly $Z_S = X_{S,S}$ for all S . When $A \not\subseteq S$, we shall use the convention $Y_{S,A} = \emptyset$. Note, that in general the sets $Y_{S,A}$ need not be closed under taking boundary.

Proposition 5.2. *The sets $X_{S,A}$ are closed under taking boundary, hence form simplicial subcomplexes of $P(\bar{r})$.*

Proof. Let $\sigma = ((W_0, G_0), \dots, (W_t, G_t))$ be a simplex in $X_{S,A}$, and assume $\tau \subset \sigma$. By Proposition 3.2 there exists $T \subseteq A(\sigma)$, such that $\tau = \Gamma_T(\sigma)$. By Proposition 2.18 it is enough to consider the case $|T| = 1$, so assume $T = \{p\}$, and let $\tau = ((\widetilde{W}_0, \widetilde{G}_0), \dots, (\widetilde{W}_t, \widetilde{G}_t))$.

By definition of $X_{S,A}$ we have either $\sigma \in Z_S$ or $\sigma \in Y_{S,A}$. Consider first the case $\sigma \in Z_S$, so $S \subseteq G_1$. Since $\tilde{G}_1 \supseteq G_1$, we have $\tau \in Z_S$.

Now, assume $\sigma \in Y_{S,A}$. This means $W_1 \cup G_1 = S$ and $A \subseteq G_1$. Again $\tilde{G}_1 \supseteq G_1$ implies $A \subseteq \tilde{G}_1$. \square

In particular, X_S and Z_S are simplicial subcomplexes of $P(\bar{r})$, for all S . When we are dealing with several round counters, in order to avoid confusion, we shall add \bar{r} to the notations, and write $X_{S,A}(\bar{r})$, $X_S(\bar{r})$, $Y_{S,A}(\bar{r})$, $Z_S(\bar{r})$. We shall also let $\alpha_{S,A}(\bar{r})$ denote the inclusion map

$$\alpha_{S,A}(\bar{r}) : X_{S,A}(\bar{r}) \hookrightarrow P(\bar{r}).$$

5.2. The strata of the canonical decomposition as immediate snapshot complexes.

Proposition 5.3. *Assume $A \subseteq S \subseteq \text{act } \bar{r}$, then there exists a simplicial isomorphism*

$$\gamma_{S,A}(\bar{r}) : X_{S,A}(\bar{r}) \xrightarrow{\sim} P(\bar{r}_{S,A}).$$

Proof. We start by considering the case $A = \emptyset$. Pick an arbitrary simplex $\sigma = ((W_0, G_0), \dots, (W_t, G_t))$ belonging to X_S . By the construction of X_S , we either have $W_1 \cup G_1 = S$, or $S \subseteq G_1$. If $W_1 \cup G_1 = S$, then set

$$\gamma_S(\sigma) := \begin{array}{|c|c|c|c|c|} \hline W_0 \setminus G_1 & W_2 & \dots & W_t & \\ \hline G_0 \cup G_1 & G_2 & \dots & G_t & \\ \hline \end{array},$$

else $S \subseteq G_1$, in which case we set

$$\gamma_S(\sigma) := \begin{array}{|c|c|c|c|c|} \hline W_0 \setminus S & W_1 & W_2 & \dots & W_t \\ \hline G_0 \cup S & G_1 \setminus S & G_2 & \dots & G_t \\ \hline \end{array}.$$

Reversely, assume $\tau = ((V_0, H_0), \dots, (V_t, H_t))$ is a simplex of $P(\bar{r}_S)$. Note, that in any case, we have $S \subseteq V_0 \cup H_0$. If $V_0 \cap S \neq \emptyset$, we set

$$\rho_S(\tau) := \begin{array}{|c|c|c|c|c|} \hline V_0 \cup (H_0 \cap S) & V_0 \cap S & V_1 & \dots & V_t \\ \hline H_0 \setminus (H_0 \cap S) & H_0 \cap S & H_1 & \dots & H_t \\ \hline \end{array},$$

else $S \subseteq H_0$, and we set

$$\rho_S(\tau) := \begin{array}{|c|c|c|c|c|} \hline V_0 \cup S & V_1 & V_2 & \dots & V_t \\ \hline H_0 \setminus S & H_1 \cup S & H_2 & \dots & H_t \\ \hline \end{array}.$$

It is immediate that γ_S and ρ_S preserve supports, $A(-)$, $G(-)$, and hence also the dimension. Furthermore, we can see what happens with the cardinalities of the traces. For all elements p which do not belong to S , the cardinalities of their traces are preserved. For all elements in S , the map γ_S decreases the cardinality of the trace, whereas, the map ρ_S increases it. It follows that γ_S and ρ_S are well-defined as dimension-preserving maps between sets of simplices.

To see that γ_S preserves boundaries, pick a top-dimensional simplex $\sigma = (W_0, S, W_1, \dots, W_t)$ in X_S and ghost the set T . Assume first $S \not\subseteq T$. In this case not all elements in S are ghosted. Assume now that $S \subseteq T$. This implies that γ_S is well-defined as a simplicial map. Finally, a direct verification shows that the maps γ_S and ρ_S are inverses of each other, hence they are simplicial isomorphisms.

Let us now consider the case when A is arbitrary. The simplicial complex $X_{S,A}$ is a subcomplex of X_S consisting of all simplices σ satisfying the additional condition $A \subseteq G_1$. The image $\gamma_S(X_{S,A})$ consists of all $\tau = ((V_0, H_0), \dots, (V_t, H_t))$

in $P(\bar{r}_{S,A})$ satisfying $A \subseteq H_0$. The map $\Xi : \gamma_S(X_{S,A}) \rightarrow P(\bar{r}_{S,A})$, taking τ to $((V_0, H_0 \setminus A), (V_1, H_1), \dots, (V_t, H_t))$, is obviously a simplicial isomorphism, hence the composition $\gamma_{S,A} = \Xi \circ \gamma_S : X_{S,A} \rightarrow P(\bar{r}_{S,A})$ is a simplicial isomorphism as well. \square

Note that, in particular,

$$\gamma_{A,A}(\sigma) = \begin{array}{|c|c|c|c|c|} \hline W_0 \setminus A & W_1 & W_2 & \dots & W_t \\ \hline G_0 & G_1 \setminus A & G_2 & \dots & G_t \\ \hline \end{array}.$$

Proposition 5.4. *Assume \bar{r} is an arbitrary round counter, and $S, A \subset \text{act } \bar{r}$, such that $S \cap A = \emptyset$, then the following diagram commutes*

$$(5.1) \quad \begin{array}{ccc} X_{A,A}(\bar{r}) & \xleftarrow{i} & X_{S \cup A, A}(\bar{r}) \\ \gamma_{A,A}(\bar{r}) \downarrow & & \searrow \gamma_{S \cup A, A}(\bar{r}) \\ P(\bar{r} \setminus A) & \xleftarrow{\alpha_S(\bar{r} \setminus A)} & X_S(\bar{r} \setminus A) \xrightarrow{\gamma_S(\bar{r} \setminus A)} P(\bar{r}_{S,A}), \end{array}$$

where i denotes the strata inclusion map.

Proof. To start with, note that $\bar{r}_{S,A} = (\bar{r} \downarrow S) \setminus A = (\bar{r} \downarrow (S \cup A)) \setminus A$, so the diagram (5.1) is well-defined. To see that it is commutative, pick an arbitrary $\sigma = ((W_0, G_0), \dots, (W_t, G_t))$. We know that either $A \subseteq G_1$ and $W_1 \cup G_1 = S \cup A$, or $A \cup S \subseteq G_1$. On one hand, we have

$$(\gamma_{A,A}(\bar{r}) \circ i)(\sigma) = \begin{array}{|c|c|c|c|c|} \hline W_0 \setminus A & W_1 & W_2 & \dots & W_t \\ \hline G_0 & G_1 \setminus A & G_2 & \dots & G_t \\ \hline \end{array}.$$

On the other hand, we have

$$\gamma_{S \cup A, A}(\bar{r})(\sigma) = \begin{cases} \begin{array}{|c|c|c|c|} \hline W_0 \setminus G_1 & W_2 & \dots & W_t \\ \hline G_0 \cup G_1 \setminus A & G_2 & \dots & G_t \\ \hline \end{array}, & \text{if } A \subseteq G_1, W_1 \cup G_1 = S \cup A; \\ \begin{array}{|c|c|c|c|c|} \hline W_0 \setminus (S \cup A) & W_1 & W_2 & \dots & W_t \\ \hline G_0 \cup S & G_1 \setminus (S \cup A) & G_2 & \dots & G_t \\ \hline \end{array}, & \text{if } A \cup S \subseteq G_1. \end{cases}$$

Applying $\gamma_S(\bar{r} \setminus A)^{-1}$ we can verify that $\gamma_{A,A}(\bar{r}) \circ i = \alpha_S(\bar{r} \setminus A) \circ \gamma_S(\bar{r} \setminus A)^{-1} \circ \gamma_{S \cup A, A}(\bar{r})$. \square

Corollary 5.5. *For any $A \subseteq \text{act } \bar{r}$, we have*

$$(5.2) \quad X_{A,A}(\bar{r}) = \bigcup_{\emptyset \neq S \subseteq \text{act } \bar{r} \setminus A} X_{S \cup A, A}(\bar{r}) = \bigcup_{A \subseteq S \subseteq \text{act } \bar{r}} X_{S,A}(\bar{r}).$$

Proof. Since $P(\bar{r} \setminus A) = \bigcup_{\emptyset \neq S \subseteq \text{act } \bar{r} \setminus A} X_S(\bar{r} \setminus A)$, the equation (5.2) is an immediate consequence of the commutativity of the diagram (5.1). \square

5.3. The incidence structure of the canonical decomposition.

Clearly, $P(\bar{r}) = \bigcup_S X_S$. We describe here the complete combinatorics of intersecting these pieces.

Proposition 5.6. *For all pairs of subsets $A \subseteq S \subseteq \text{supp } \bar{r}$ and $B \subseteq T \subseteq \text{supp } \bar{r}$ we have: $X_{S,A} \subseteq X_{T,B}$ if and only if at least one of the following two conditions is satisfied:*

- $S = T$ and $B \subseteq A$,
- $T \subseteq A$.

We remark that it can actually happen that both conditions in Proposition 5.6 are satisfied. This happens exactly when $S = T = A$.

Proof of Proposition 5.6. First we show that $T \subseteq A$ implies $X_{S,A} \subseteq X_{T,B}$. Take $\sigma \in X_{S,A}$. If $\sigma \in Z_S$, then we have the following chain of implications: $S \subseteq G_1 \Rightarrow A \subseteq G_1 \Rightarrow T \subseteq G_1 \Rightarrow \sigma \in Z_T$. If, on the other hand, $\sigma \in Y_{S,A}$, we also have $A \subseteq G_1$, implying $T \subseteq G_1$, hence $\sigma \in Z_T$.

Next we show that if $S = T$ and $B \subseteq A$, then $X_{S,A} \subseteq X_{S,B}$. Clearly, we just need to show that $Y_{S,A} \subseteq X_{S,B}$. Take $\sigma \in Y_{S,A}$, then we have the following chain of implications:

$$\begin{cases} R_1 = S \\ A \subseteq G_1 \end{cases} \Rightarrow \begin{cases} R_1 = T \\ B \subseteq G_1 \end{cases} \Rightarrow \sigma \in Y_{T,B}.$$

This proves the *if* part of the proposition.

To prove the *only if* part, assume $X_{S,A} \subseteq X_{T,B}$. If $S \neq A$, set

$$\tau := \begin{array}{|c|c|c|c|c|} \hline \text{supp } \bar{r} & S \setminus A & p_1 & \dots & p_t \\ \hline \emptyset & A & \emptyset & \dots & \emptyset \\ \hline \end{array},$$

else $S = T$, and we set

$$\tau := \begin{array}{|c|c|c|c|c|} \hline \text{supp } \bar{r} & p_1 & p_2 & \dots & p_t \\ \hline \emptyset & S & \emptyset & \dots & \emptyset \\ \hline \end{array},$$

where in both cases p_1, \dots, p_t is a sequence of elements from $\text{supp } \bar{r} \setminus A$, with each element p occurring $\bar{r}(p)$ times. Clearly, in the first case, $\tau \in Y_{S,A}$, and in the second case $\tau \in Z_S$, hence $\tau \in X_{T,B} = Z_T \cup Y_{T,B}$. This means that either $T \subseteq A$, or $S = T$ and $B \subseteq A$. \square

Lemma 5.7. Assume $A \subseteq S \subseteq \text{supp } \bar{r}$ and $B \subseteq T \subseteq \text{supp } \bar{r}$. We have

- (1) $Z_S \cap Z_T = Z_{S \cup T}$,
- (2) $Y_{S,A} \cap Z_T = Y_{S, A \cup T}$,
- (3) $Y_{S,A} \cap Y_{T,B} = \begin{cases} Y_{S, A \cup B}, & \text{if } S = T, \\ \emptyset, & \text{otherwise.} \end{cases}$

Proof. To show (1), pick $\sigma \in Z_S \cap Z_T$. We have $S \subseteq G_1$ and $T \subseteq G_1$, hence $S \cup T \subseteq G_1$, and so $\sigma \in Z_{S \cup T}$.

To show (2), pick $\sigma \in Y_{S,A} \cap Z_T$. We have $R_1 = S$, $A \subseteq G_1$, and $T \subseteq G_1$. It follows that $R_1 = S$ and $A \cup T \subseteq G_1$, so $\sigma \in Y_{S, A \cup T}$.

Finally, to show (3), pick $\sigma \in Y_{S,A} \cap Y_{T,B}$. On one hand, $\sigma \in Y_{S,A}$ means $R_1 = S$ and $A \subseteq G_1$, on the other hand, $\sigma \in Y_{T,B}$ means $R_1 = T$ and $B \subseteq G_1$. We conclude that $Y_{S,A} \cap Y_{T,B} = \emptyset$ if $S \neq T$. Otherwise, we have $R_1 = S = T$ and $A \cup B \subseteq G_1$, so $\sigma \in Y_{S, A \cup B}$. \square

Proposition 5.8. For all pairs of subsets $A \subseteq S \subseteq \text{supp } \bar{r}$ and $B \subseteq T \subseteq \text{supp } \bar{r}$ we have the following formulae for the intersection:

$$\begin{aligned} (5.3) \quad & X_{S,A} \cap X_{T,B} = \begin{cases} X_{S, A \cup B}, & \text{if } S = T; \\ X_{T, S \cup B}, & \text{if } S \subset T; \\ Z_{S \cup T} = X_{S \cup T, S \cup T}, & \text{if } S \not\subseteq T \text{ and } T \not\subseteq S. \end{cases} \end{aligned}$$

Proof. In general, we have

$$(5.6) \quad X_{S,A} \cap X_{T,B} = (Z_S \cap Z_T) \cup (Z_S \cap Y_{T,B}) \cup (Y_{S,A} \cap Z_T) \cup (Y_{S,A} \cap Y_{T,B})$$

$$= \begin{cases} Z_{S \cup T} \cup Y_{T, S \cup B} \cup Y_{S, T \cup A} \cup Y_{S, A \cup B}, & \text{if } S = T; \\ Z_{S \cup T} \cup Y_{T, S \cup B} \cup Y_{S, T \cup A}, & \text{otherwise.} \end{cases}$$

Assume first that $S = T$. In this case $Y_{T, S \cup B} = Y_{S, T \cup A} = Z_S$, hence the equation (5.6) translates to $X_{S,A} \cap X_{T,B} = Z_S \cup Y_{S, A \cup B} = X_{S, A \cup B}$.

Let us now consider the case $S \subset T$. We have $Y_{S, T \cup A} = \emptyset$, hence (5.6) translates to $X_{S,A} \cap X_{T,B} = Z_T \cup Y_{T, S \cup B} = X_{T, S \cup B}$.

Finally, assume $S \not\subset T$ and $T \not\subset S$. Then $Y_{T, S \cup B} = Y_{S, T \cup A} = \emptyset$, hence (5.6) says $X_{S,A} \cap X_{T,B} = Z_{S \cup T}$. \square

For convenience we record the following special cases of Proposition 5.8.

Corollary 5.9. *For $S \neq T$ we have*

$$X_S \cap X_T = \begin{cases} X_{T,S}, & \text{if } S \subset T, \\ Z_{S \cup T}, & \text{otherwise,} \end{cases}$$

$$(5.7) \quad X_S \cap Z_T = Z_{S \cup T}.$$

Proof. The first formula is a simple substitution of $A = B = \emptyset$ in (5.4) and (5.5). To see (5.7), substitute $A = \emptyset$, $B = T$ in (5.4) to obtain

$$X_{S,\emptyset} \cap X_{T,T} = \begin{cases} X_{T,S \cup T}, & \text{if } S \subset T \\ Z_{S \cup T}, & \text{otherwise} \end{cases} = \begin{cases} Z_T, & \text{if } S \subset T \\ Z_{S \cup T}, & \text{otherwise} \end{cases} = Z_{S \cup T}. \quad \square$$

Remark 5.10. *Corollary 5.9 implies that every stratum $X_{S,A}$ can be represented as an intersection of two strata of the type X_S , with only exception provided by the strata $X_{S,S}$, when $|S| = 1$.*

Corollary 5.11. *Assume $S_1, \dots, S_t \subseteq [n]$, such that $S_1 \not\subset S_i$, for all $i = 2, \dots, t$. The following two cases describe the intersection $X_{S_1} \cap \dots \cap X_{S_t}$:*

- (1) *if $S_1 \supset S_i$, for all $i = 2, \dots, t$, then $X_{S_1} \cap \dots \cap X_{S_t} = X_{S_1, S_2 \cup \dots \cup S_t}$;*
- (2) *if there exists $2 \leq i \leq t$, such that $S_1 \not\supset S_i$, then $X_{S_1} \cap \dots \cap X_{S_t} = Z_{S_1 \cup S_2 \cup \dots \cup S_t} = X_{S_1 \cup S_2 \cup \dots \cup S_t, S_1 \cup S_2 \cup \dots \cup S_t}$.*

Proof. Assume first that $S_1 \supset S_i$, for all $i = 2, \dots, t$. By iterating (5.4) we get

$$\begin{aligned} X_{S_1} \cap \dots \cap X_{S_t} &= X_{S_1, \emptyset} \cap X_{S_2, \emptyset} \cap \dots \cap X_{S_t, \emptyset} = X_{S_1, S_2} \cap X_{S_3, \emptyset} \cap \dots \cap X_{S_t, \emptyset} \\ &= X_{S_1, S_2 \cup S_3} \cap X_{S_4, \emptyset} \cap \dots \cap X_{S_t, \emptyset} = \dots = X_{S_1, S_2 \cup \dots \cup S_t}. \end{aligned}$$

This proves (1).

To show (2), we can assume without loss of generality, that $S_2 \not\subset S_1$. By (5.5) we have $X_{S_1} \cap X_{S_2} = Z_{S_1 \cup S_2}$. By iterating (5.7) we get

$$Z_{S_1 \cup S_2} \cap X_{S_3} \cap \dots \cap X_{S_t} = Z_{S_1 \cup S_2 \cup S_3} \cap X_{S_4} \cap \dots \cap X_{S_t} = X_{S_1 \cup S_2 \cup \dots \cup S_t},$$

which finishes the proof. \square

5.4. The boundary of the immediate snapshot complexes and its canonical decomposition.

Definition 5.12. Let \bar{r} be an arbitrary round counter, and assume $V \subset \text{supp } \bar{r}$. We define $B_V(\bar{r})$ to be the simplicial subcomplex of $P(\bar{r})$ consisting of all simplices $\sigma = ((W_0, G_0), \dots, (W_t, G_t))$, satisfying $V \subseteq G_0$.

The fact that $B_V(\bar{r})$ is a well-defined subcomplex of $P(\bar{r})$ is immediate from the definition of the ghosting operation. We shall let $\beta_V(\bar{r})$ denote the inclusion map

$$\beta_V(\bar{r}) : B_V(\bar{r}) \hookrightarrow P(\bar{r}).$$

Proposition 5.13. For an arbitrary round counter \bar{r} , and any $V \subset \text{supp } \bar{r}$, the map $\delta_V(\bar{r})$ given by

$$\delta_V(\bar{r}) : ((W_0, G_0), \dots, (W_t, G_t)) \mapsto ((W_0, G_0 \setminus V), \dots, (W_t, G_t))$$

is a simplicial isomorphism between simplicial complexes $B_V(\bar{r})$ and $P(\bar{r} \setminus V)$.

Proof. The map $\delta_V(\bar{r})$ is simplicial, and it has a simplicial inverse which adds V to G_0 . \square

Given an arbitrary round counter \bar{r} , $A \subseteq S \subseteq \text{act } \bar{r}$, and $V \subset \text{supp } \bar{r}$, such that $S \cap V = \emptyset$, we set

$$X_{S,A,V}(\bar{r}) := X_{S,A}(\bar{r}) \cap B_V(\bar{r}).$$

We can use the notational convention $B_\emptyset(\bar{r}) = P(\bar{r})$, which is consistent with Definition 5.12. In this case we get $X_{S,A,\emptyset}(\bar{r}) = X_{S,A}(\bar{r})$, fitting well with the previous notations.

Proposition 5.14. Assume \bar{r} is an arbitrary round counter, $V \subset \text{supp } \bar{r}$, $A \subseteq S \subseteq \text{act } \bar{r}$, and $V \cap S = \emptyset$. Then there exist simplicial isomorphisms φ and ψ making the following diagram commute:

$$(5.8) \quad \begin{array}{ccccc} P(\bar{r}) & \xleftarrow{\alpha} & X_{S,A}(\bar{r}) & \overset{\gamma}{\rightsquigarrow} & P(\bar{r}_{S,A}) \\ \uparrow \beta & & \uparrow j & & \uparrow \beta \\ B_V(\bar{r}) & \xleftarrow{i} & X_{S,A,V}(\bar{r}) & \overset{\varphi}{\rightsquigarrow} & B_V(\bar{r}_{S,A}) \\ \downarrow \delta & & \downarrow \psi & & \downarrow \delta \\ P(\bar{r} \setminus V) & \xleftarrow{\alpha} & X_{S,A}(\bar{r} \setminus V) & \overset{\gamma}{\rightsquigarrow} & P(\bar{r}_{S \cup V, A \cup V}), \end{array}$$

where i and j denote inclusion maps.

Proof. Note that $X_{S,A,V}(\bar{r})$ consists of all simplices $\sigma = ((W_0, G_0), \dots, (W_t, G_t))$, such that $V \subseteq G_0$, $A \subseteq G_1$, and either $W_1 \cup G_1 = S$, or $S \subseteq G_1$. The fact that V and S are disjoint ensures that these conditions do not contradict each other. We let φ be the restriction of $\gamma_{S,A}(\bar{r}) : X_{S,A}(\bar{r}) \rightarrow P(\bar{r}_{S,A})$ to $X_{S,A,V}(\bar{r})$. Furthermore, we let ψ be the restriction of $\delta_V(\bar{r}) : B_V(\bar{r}) \rightarrow P(\bar{r} \setminus V)$ to $X_{S,A,V}(\bar{r})$. \square

The diagram (5.8) means that we can naturally think about $X_{S,A,V}(\bar{r})$ both as $X_{S,A}(\bar{r} \setminus V)$ as well as $B_V(\bar{r}_{S,A})$, or abusing notations we write $B_V \cap X_{S,A} = X_{S,A}(B_V) = B_V(X_{S,A})$.

Proposition 5.15. *Assume $B \subseteq A \subseteq S \subseteq \text{act } \bar{r}$, then the following diagram commutes*

$$(5.9) \quad \begin{array}{ccccc} X_{S,B}(\bar{r}) & \xleftarrow{i} & X_{S,A}(\bar{r}) & & \\ \downarrow \gamma_{S,B}(\bar{r}) & & \searrow \gamma_{S,A}(\bar{r}) & & \\ P(\bar{r}_{S,B}) & \xleftarrow{\beta_{A \setminus B}(\bar{r}_{S,B})} & B_{A \setminus B}(\bar{r}_{S,B}) & \xrightarrow{\delta_{A \setminus B}(\bar{r}_{S,B})} & P(\bar{r}_{S,A}) \end{array}$$

where i denotes the inclusion map.

Proof. Take $\sigma = ((W_0, G_0), \dots, (W_t, G_t)) \in X_{S,A}(\bar{r})$. On one hand we have

$$(\gamma_{S,B}(\bar{r}) \circ i)(\sigma) = \begin{cases} \begin{array}{|c|c|c|c|} \hline W_0 \setminus G_1 & W_2 & \dots & W_t \\ \hline G_0 \cup G_1 \setminus B & G_2 & \dots & G_t \\ \hline \end{array}, & \text{if } W_1 \cup G_1 = S, A \subseteq G_1; \\ \begin{array}{|c|c|c|c|c|} \hline W_0 \setminus S & W_1 & W_2 & \dots & W_t \\ \hline G_0 \cup S \setminus B & G_1 \setminus S & G_2 & \dots & G_t \\ \hline \end{array}, & \text{if } S \subseteq G_1. \end{cases}$$

On the other hand, we have

$$(\gamma_{S,A}(\bar{r}))(\sigma) = \begin{cases} \begin{array}{|c|c|c|c|} \hline W_0 \setminus G_1 & W_2 & \dots & W_t \\ \hline G_0 \cup G_1 \setminus A & G_2 & \dots & G_t \\ \hline \end{array}, & \text{if } W_1 \cup G_1 = S, A \subseteq G_1; \\ \begin{array}{|c|c|c|c|c|} \hline W_0 \setminus S & W_1 & W_2 & \dots & W_t \\ \hline G_0 \cup S \setminus A & G_1 \setminus S & G_2 & \dots & G_t \\ \hline \end{array}, & \text{if } S \subseteq G_1. \end{cases}$$

Since applying $\delta_{A \setminus B}(\bar{r}_{S,B})^{-1}$ will add $A \setminus B$ to $G_0 \cup G_1 \setminus A$, resp. $G_0 \cup S \setminus A$, above and $A \subseteq S$, $A \subseteq G_1$, we conclude that

$$(\gamma_{S,B}(\bar{r}) \circ i)(\sigma) = (\beta_{A \setminus B}(\bar{r}_{S,B}) \circ \delta_{A \setminus B}(\bar{r}_{S,B})^{-1} \circ \gamma_{S,A}(\bar{r}))(\sigma).$$

Which is the same as to say that the diagram (5.9) commutes. \square

5.5. The combinatorial structure of the complexes $P(\chi_{A,B})$.

Let us analyze the simplicial structure of $P(\chi_{A,B})$. Set $k := |A| - 1$ and $m := |B|$. By (4.2) the simplicial complex $P(\chi_{A,B})$ is isomorphic to the m -fold suspension of $P(\chi_A)$. On the other hand, we saw in subsection 4.5 that $P(\chi_A)$ is isomorphic to the standard chromatic subdivision of Δ^k . The simplices of the m -fold suspension of $\chi(\Delta^k)$ (which is of course homeomorphic to Δ^{m+k}) are indexed by tuples $(S, (B_1, \dots, B_t)(C_1, \dots, C_t))$, where S is any subset of B , and the sets $B_1, \dots, B_t, C_1, \dots, C_t$ satisfy the same conditions as in the combinatorial description of the simplicial structure of $\chi(\Delta^k)$. In line with (4.6), the simplicial isomorphism between $P(\chi_{A,B})$ and the m -fold suspension of $\chi(\Delta^k)$ can be explicitly given by

$$(S, (B_1, \dots, B_t)(C_1, \dots, C_t)) \mapsto \begin{array}{|c|c|c|c|} \hline W_0 & C_1 & \dots & C_t \\ \hline (A \cup B) \setminus W_0 & B_1 \setminus C_1 & \dots & B_t \setminus C_t \\ \hline \end{array}$$

where $W_0 = S \cup B_1 \cup \dots \cup B_t$. In particular, up to the simplicial isomorphism, the complex $P(\chi_{A,B})$ depends only on m and k .

In analogy with subsection 4.5 the simplices of $P(\chi_{A,B})$ are indexed by all witness structures $\sigma = ((W_0, G_0), \dots, (W_t, G_t))$ satisfying the following conditions:

- (1) $W_0 \cup G_0 = A \cup B$;
- (2) $W_0 \cap A = W_1 \cup \dots \cup W_t \cup G_1 \cup \dots \cup G_t$;

(3) the sets $W_1, \dots, W_t, G_1, \dots, G_t$ are disjoint.

It was shown in [Ko12] that there is a homeomorphism

$$\tau_A : P(\chi_A) \xrightarrow[\cong]{} \Delta^A,$$

such that for any $C \subseteq A$ the following diagram commutes

$$(5.10) \quad \begin{array}{ccccc} P(\chi_A) & \xleftarrow{\beta_{A \setminus C}(\chi_A)} & B_{A \setminus C}(\chi_A) & \xrightarrow{\delta_{A \setminus C}(\chi_A)} & P(\chi_C) \\ \tau_A \downarrow \cong & & & & \nearrow \tau_C \cong \\ \Delta^A & \xleftarrow{i} & \Delta^C & & \end{array}$$

where $i : \Delta^C \hookrightarrow \Delta^A$ is the standard inclusion map. In general, given a pair of sets (A, B) , we take the $|B|$ -fold suspension of the map τ_A to produce a homeomorphism

$$\tau_{A,B} : P(\chi_{A,B}) \xrightarrow[\cong]{} \Delta^{A \cup B}.$$

Definition 5.16. When $A \cup B = C \cup D$, we set

$$\tau(\chi_{A,B}, \chi_{C,D}) := \tau_{C,D}^{-1} \circ \tau_{A,B},$$

clearly, we get a homeomorphism $\tau(\chi_{A,B}, \chi_{C,D}) : P(\chi_{A,B}) \xrightarrow[\cong]{} P(\chi_{C,D})$.

We know that this map is a simplicial isomorphism when restricted to $B_S(\chi_{A,B})$, for all $S \subseteq (A \cap C) \cup (B \cap D)$, i.e., we have the following commutative diagram

$$(5.11) \quad \begin{array}{ccc} B_S(\chi_{A,B}) & \xrightarrow{\tau(\chi_{A,B}, \chi_{C,D})} & B_S(\chi_{C,D}) \\ \downarrow \beta_S(\chi_{A,B}) & & \downarrow \beta_S(\chi_{C,D}) \\ P(\chi_{A,B}) & \xrightarrow[\cong]{\tau(\chi_{A,B}, \chi_{C,D})} & P(\chi_{C,D}) \end{array}$$

When $C \subseteq A$, we have $B \subseteq D$, so the condition for S becomes $S \subseteq B \cup C$. Furthermore, if in addition $T = E \cup F$, we have

$$\tau(\chi_{A_1, B_1}, \chi_{A_2, B_2}) \circ \tau(\chi_{A_2, B_2}, \chi_{A_3, B_3}) = \tau(\chi_{A_1, B_1}, \chi_{A_3, B_3}).$$

When $A \subseteq C \cup D$, the identity (2.1) implies that we have a simplicial isomorphism

$$\beta_V(\chi_{C,D}) : B_V(\chi_{C,D}) \xrightarrow[\cong]{} P(\chi_{C \setminus A, D \setminus A}).$$

Furthermore, when $S \subseteq C$, the identity (2.4) implies that we have a simplicial isomorphism

$$X_S(\chi_{C,D}) : \gamma_S(\chi_{C,D}) \xrightarrow[\cong]{} P(\chi_{C \setminus S, D \cup S}).$$

Proposition 5.17. Assume $A \cup B = C \cup D$ and $V \subseteq A \cup B$, then the following diagram commutes

$$(5.12) \quad \begin{array}{ccccc} P(\chi_{A,B}) & \xleftarrow{\beta_V(\chi_{A,B})} & B_V(\chi_{A,B}) & \xrightarrow{\delta_V(\chi_{A,B})} & P(\chi_{A,B} \setminus V) \\ \tau(\chi_{A,B}, \chi_{C,D}) \downarrow \cong & & & & \cong \downarrow \tau(\chi_{A,B} \setminus V, \chi_{C,D} \setminus V) \\ P(\chi_{C,D}) & \xleftarrow{\beta_V(\chi_{C,D})} & B_V(\chi_{C,D}) & \xrightarrow{\delta_V(\chi_{C,D})} & P(\chi_{C,D} \setminus V) \end{array}$$

Proof. Consider the diagram on Figure 5.1. Both the upper and the lower part of this diagram are versions of (5.10), hence, they commute. Together, they form the diagram (5.12). \square

$$\begin{array}{ccccc}
 P(\chi_{A,B}) & \xleftarrow{\beta_V(\chi_{A,B})} & B_V(\chi_{A,B}) & \xrightarrow{\delta_V(\chi_{A,B})} & P(\chi_{A,B} \setminus V) \\
 \tau_{A,B} \downarrow \cong & & & \nearrow \cong & \\
 \Delta^{A \cup B} & \xleftarrow{\quad} & \Delta^{A \cup B \setminus V} & & \\
 \tau_{C,D} \uparrow \cong & & & \nwarrow \cong & \\
 P(\chi_{C,D}) & \xleftarrow{\beta_V(\chi_{C,D})} & B_V(\chi_{C,D}) & \xrightarrow{\delta_V(\chi_{C,D})} & P(\chi_{C,D} \setminus V)
 \end{array}$$

$\tau_{A \setminus V, B \setminus V}$ and $\tau_{C \setminus V, D \setminus V}$ are also indicated on the right side of the diagram.

FIGURE 5.1.

6. TOPOLOGY OF THE IMMEDIATE SNAPSHOT COMPLEXES

6.1. Immediate snapshot complexes are collapsible pseudomanifolds.

Before proceeding to the main result, that the immediate snapshot complexes are simplicially homeomorphic simplices, we give short proofs of the facts that these complexes are pseudomanifolds, that they are contractible topological spaces, and stronger, that they are collapsible simplicial complexes.

We start by showing that $P(\bar{r})$ is a pseudomanifold.

Definition 6.1. Let K be a pure simplicial complex of dimension n . Two n -simplices of K are said to be **strongly connected** if there is a sequence of n -simplices so that each pair of consecutive simplices has a common $(n-1)$ -dimensional face. The complex K is said to be **strongly connected** if any two n -simplices of K are strongly connected.

Clearly, being strongly connected is an equivalence relation on the set of all n -simplices.

Proposition 6.2. For an arbitrary round counter \bar{r} , the simplicial complex $P(\bar{r})$ is strongly connected.

Proof. Set $n := |\text{supp } \bar{r}| - 1$. Proposition 4.1 says that $P(\bar{r})$ is a pure simplicial complex of dimension n . We now use induction on $|\bar{r}|$. If $|\bar{r}| = 0$, or more generally, if $|\text{act } \bar{r}| \leq 1$, then $P(\bar{r})$ is just a single simplex, so it is trivially strongly connected.

Assume $|\text{act } \bar{r}| \geq 2$, and consider the canonical decomposition of $P(\bar{r})$. By Proposition 5.3, the simplicial complex $X_S(\bar{r})$ is isomorphic to $P(\bar{r}_S)$, for all $S \subseteq \text{act } \bar{r}$. Since $|\bar{r}_S| = |\bar{r}| - |S| < |\bar{r}|$, and $\text{supp } \bar{r}_S = \text{supp } \bar{r}$, we conclude that $X_S(\bar{r})$ is a pure simplicial complex of dimension n , which is strongly connected by the induction assumption. Thus, any pair of n -simplices belonging to the same subcomplex $X_S(\bar{r})$ is strongly connected.

Pick now any $p \in \text{act } \bar{r}$, and any $S \subseteq \text{act } \bar{r}$, such that $p \in S$, and consider the subcomplex $X_{S,p}(\bar{r}) = X_S(\bar{r}) \cap X_p(\bar{r})$. According to Proposition 5.3, this subcomplex is isomorphic to $P(\bar{r}_{S,p})$, in particular, it is non-empty. Take any $(n-1)$ -simplex τ in $X_{S,p}(\bar{r})$. By induction assumptions for $X_S(\bar{r})$ and $X_p(\bar{r})$, there exist n -simplices $\sigma_1 \in X_S(\bar{r})$, and $\sigma_2 \in X_p(\bar{r})$, such that $\tau \in \partial\sigma_1$ and $\tau \in \partial\sigma_2$. This means, that σ_1 and σ_2 are strongly connected. Since being strongly connected is an equivalence relation, any two n -simplices from $X_S(\bar{r})$ and $X_p(\bar{r})$ are strongly connected. This includes the case $S = \text{act } \bar{r}$, implying that any pair of n -simplices in $P(\bar{r})$ is strongly connected, so $P(\bar{r})$ itself is strongly connected. \square

Definition 6.3. We say that a strongly connected pure simplicial complex K is a **pseudomanifold** if each $(n-1)$ -simplex of K is a face of precisely one or two

n -simplices of K . The $(n-1)$ -simplices of K which are faces of precisely one n -simplex of K form a simplicial subcomplex of K , called the **boundary** of K , and denoted ∂K .

Proposition 6.4. *For an arbitrary round counter \bar{r} , the simplicial complex $P(\bar{r})$ is a pseudomanifold, such that $\partial P(\bar{r}) = \cup_{p \in \text{supp } \bar{r}} B_p(\bar{r})$, i.e., the subcomplex $\partial P(\bar{r})$ consists of all simplices $\sigma = ((W_0, G_0), \dots, (W_t, G_t))$, such that $G_0 \neq \emptyset$.*

Proof. By Proposition 6.2 we already know that $P(\bar{r})$ is strongly connected. Set again $n := |\text{supp } \bar{r}| - 1$, and let $\tau = ((W_0, G_0), \dots, (W_t, G_t))$ be an arbitrary $(n-1)$ -simplex of $P(\bar{r})$. Note that $\text{codim } \tau = |G_0| + \dots + |G_t|$, hence $\text{codim } \tau = 1$ implies that there exist $0 \leq k \leq t$, and $p \in \text{supp } \bar{r}$, such that

$$G_i = \begin{cases} \{p\}, & \text{if } i = k; \\ \emptyset, & \text{if } i \neq k. \end{cases}$$

Set $m := r(p) + 1$. Consider

$$\sigma_1 = (W_0, \dots, W_k - 1, W_k \cup \{p\}, W_{k+1}, \dots, W_t, \underbrace{p, \dots, p}_m),$$

and if $k \geq 1$, consider also

$$\sigma_2 = (W_0, \dots, W_k - 1, p, W_k, \dots, W_t, \underbrace{p, \dots, p}_m).$$

Obviously, $\Gamma(\sigma_1, p) = \Gamma(\sigma_2, p) = \tau$, so $\tau \in \partial \sigma_1$ and $\tau \in \partial \sigma_2$. Furthermore, the definition of the ghosting construction implies that these are the only options to find σ , such that $\Gamma(\sigma, p) = \tau$.

We conclude that $P(\bar{r})$ is a pseudomanifold, whose boundary is a union of the $(n-1)$ simplices $\tau = ((W_0, G_0), \dots, (W_t, G_t))$, such that $W_0 \neq \emptyset$, so then the subcomplex $\partial P(\bar{r})$ consists of all simplices $\sigma = ((W_0, G_0), \dots, (W_t, G_t))$, such that $G_0 \neq \emptyset$. \square

Consider now a quite general situation, where X is an arbitrary topological space, and $\{X_i\}_{i \in I}$ is a finite family of subspace of X covering X , that is I is finite and $X = \cup_{i \in I} X_i$.

Definition 6.5. ([Ko07, Definition 15.14]). *The **nerve complex** \mathcal{N} of a covering $\{X_i\}_{i \in I}$ is a simplicial complex whose vertices are indexed by I , and a subset of vertices $J \subseteq I$ spans a simplex if and only if the intersection $\cap_{i \in J} X_i$ is not empty.*

The nerve complex can be useful because of the following fact.

Lemma 6.6. (Nerve Lemma, [Ko07, Theorem 15.21, Remark 15.22]). *Assume K is a simplicial complex, covered by a family of subcomplexes $\mathcal{K} = \{K_i\}_{i \in I}$, such that $\cap_{i \in J} K_i$ is empty or contractible for all $J \subseteq I$, then K is homotopy equivalent to the nerve complex $\mathcal{N}(\mathcal{K})$.*

Corollary 6.7. *For an arbitrary round counter \bar{r} , the simplicial complex $P(\bar{r})$ is contractible.*

Proof. We use induction on $|\bar{r}|$. If $|\bar{r}| = 0$, then $P(\bar{r})$ is just a simplex, hence contractible. We assume that $|\bar{r}| \geq 1$, and view the canonical decomposition $P(\bar{r}) = \cup_{S \subseteq \text{act } \bar{r}} X_S(\bar{r})$ as a covering of $P(\bar{r})$. By Proposition 5.3, Corollary 5.11, and the induction assumption, all the intersections of the subcomplexes $X_S(\bar{r})$ with each

other are either empty or contractible. This means, that we can apply the Nerve Lemma 6.6, with $K = P(\bar{r})$, $I = 2^{\text{act } \bar{r}} \setminus \{\emptyset\}$, and K_i 's are $X_S(\bar{r})$'s.

Now, by Corollary 5.11 we see that $X_{\text{act } \bar{r}} \cap X_S = X_{\text{act } \bar{r}, S} \neq \emptyset$ for all $S \subset \text{act } \bar{r}$. It follows that the nerve complex of this decomposition as a cone with apex at $\text{act } \bar{r} \in I$. Since the nerve complex is contractible, it follows from the Nerve Lemma 6.6 that $P(\bar{r})$ is contractible as well. \square

While contractibility is a property of topological spaces, there is a stronger combinatorial property called *collapsibility* which some simplicial complexes may have.

Definition 6.8. *Let K be a simplicial complex. A pair of simplices (σ, τ) of K is called an **elementary collapse** if the following conditions are satisfied:*

- τ is a maximal simplex,
- τ is the only simplex which properly contains σ .

A finite simplicial complex K is called **collapsible**, if there exists a sequence $(\sigma_1, \tau_1), \dots, (\sigma_t, \tau_t)$ of pairs of simplices of K , such that

- this sequence yields a perfect matching on the set of all simplices of K ,
- for every $1 \leq k \leq t$, the pair (σ_k, τ_k) is an elementary collapse in $K \setminus \{\sigma_1, \dots, \sigma_{k-1}, \tau_1, \dots, \tau_{k-1}\}$.

When (σ, τ) is an elementary collapse, we also say that σ is a *free* simplex.

We have shown in Proposition 6.4 that for any round counter \bar{r} the simplicial complex $P(\bar{r})$ is a pseudomanifold with boundary $\partial P(\bar{r})$. Set

$$\text{int } P(\bar{r}) := \bigcup_{\sigma \in P(\bar{r}), \sigma \notin \partial P(\bar{r})} \text{int } \sigma,$$

and, for all $A \subseteq S \subseteq \text{act } \bar{r}$, set

$$\partial X_{S,A}(\bar{r}) := \gamma_{S,A}(\bar{r})^{-1}(\partial P(\bar{r}_{S,A})), \quad \text{int } X_{S,A}(\bar{r}) := \gamma_{S,A}(\bar{r})^{-1}(\text{int } P(\bar{r}_{S,A})).$$

Proposition 6.9. *Assume \bar{r} is an arbitrary round counter, $A \subset S \subseteq \text{act } \bar{r}$, and $V \subseteq \text{supp } \bar{r} \setminus S$. The simplicial complex $\partial X_{S,A,V}(\bar{r})$ is the subcomplex of $X_{S,A,V}(\bar{r})$ consisting of all simplices*

$$\sigma = ((W_0, V), (S \setminus A, A), \dots, (W_t, G_t)).$$

Proof. Pick $\sigma \in X_{S,A,V}$, and set ρ to be the composition of the simplicial isomorphisms $X_{S,A,V}(\bar{r}) \rightarrow B_V(\bar{r}_{S,A}) \rightarrow P(\bar{r}_{S \cup V, A \cup V})$ from the commutative diagram (5.8).

Assume first that $W_1 \cup G_1 = S$, then

$$\rho(\sigma) = ((W_0 \setminus G_1, (G_0 \cup G_1) \setminus (A \cup V)), (W_2, G_2), \dots, (W_t, G_t)).$$

Clearly $\rho(\sigma) \notin \partial P(\bar{r}_{S \cup V, A \cup V})$ if and only if $(G_0 \cup G_1) \setminus (A \cup V) = \emptyset$, i.e., $G_0 \cup G_1 \subseteq A \cup V$. Since we know that $A \subseteq G_1$, $V \subseteq G_0$, this means that $G_0 = V$ and $G_1 = A$, which implies $W_1 = S \setminus A$.

Assume now that $S \subseteq G_1$, then we have

$$\rho(\sigma) = ((W_0 \setminus S, (G_0 \cup S) \setminus (A \cup V)), (W_1, G_1 \setminus S), (W_2, G_2), \dots, (W_t, G_t)).$$

Here we have $\rho(\sigma) \notin \partial P(\bar{r}_{S \cup V, A \cup V})$ if and only if $(G_0 \cup S) \setminus (A \cup V) = \emptyset$, which is impossible, since $V \cap S = \emptyset$, and $A \subset V$. \square

Corollary 6.10. *The simplicial complex $P(\bar{r})$ can be decomposed as a disjoint union of the simplex $\Delta^{\text{pass } \bar{r}} = ((\text{pass } \bar{r}, \text{act } \bar{r}))$, and the sets $\text{int } X_{S,A,V}$, where (S, A, V) range over all triples satisfying $A \subset S \subseteq \text{act } \bar{r}$ and $V \subseteq \text{supp } \bar{r} \setminus S$.*

Specifically, for a simplex $\sigma \in P(\bar{r})$, $\sigma = ((W_0, G_0), \dots, (W_t, G_t))$, we have: if $t = 0$, then $\sigma \subseteq \Delta^{\text{pass } \bar{r}}$, else $\text{int } \sigma \subseteq \text{int } X_{W_1 \setminus G_1, G_1, G_0}$.

Proof. Immediate from Proposition 6.9. \square

Lemma 6.11. Assume \bar{r} is a round counter, and $p \in \text{supp } \bar{r}$, then there exists a sequence of elementary collapses reducing the simplicial complex $P(\bar{r})$ to the subcomplex $(\partial P(\bar{r})) \setminus \text{int } B_p(\bar{r})$.

Proof. The proof is again by induction on $|r|$. The case $|r| = 0$ is trivial. The simplices we need to collapse are precisely those, whose interior lies in $\text{int } P(\bar{r}) \cup \text{int } B_p(\bar{r})$. Let Σ denote the set of all strata $X_{S,A}$, where $A \subset S \subseteq \text{act } \bar{r}$, together with all strata $X_{S,A,p}$, where $A \subset S \subseteq \text{act } \bar{r}$, $p \notin S$. By Corollary 6.10, the union of the interiors of the strata in Σ is precisely $\text{int } P(\bar{r}) \cup \text{int } B_p(\bar{r})$.

We describe our collapsing as a sequence of steps. At each step we pick a certain pair of strata (Y, X) , where $Y \subset X$, which we must “collapse”. Then, we use one of the previous results to show that as a simplicial pair (Y, X) is isomorphic to $(B_t(\bar{r}'), P(\bar{r}'))$, for some round counter \bar{r}' , such that $|\bar{r}'| < |\bar{r}|$. By induction assumption this means that there is a sequence of simplicial collapses which removes $\text{int } X \cup \text{int } Y$. Finally, we order these pairs of strata with disjoint interiors $(Y_1, X_1), \dots, (Y_d, X_d)$ such that for every $1 \leq i \leq d$, every simplex $\sigma \in P(\bar{r})$, such that $\text{int } \sigma \subseteq \text{int } X_i \cup \text{int } Y_i$, and every $\tau \supset \sigma$, such that $\dim \tau = \dim \sigma + 1$, we have

$$(6.1) \quad \text{int } \tau \subseteq \text{int } X_1 \cup \dots \cup \text{int } X_i \cup \text{int } Y_1 \cup \dots \cup \text{int } Y_i.$$

This means, that at step i we can collapse away the pair of strata (Y_i, X_i) (i.e., collapse away those simplices whose interior is contained in $\text{int } X_i \cup \text{int } Y_i$) using the procedure given by the induction assumption, and that these elementary collapses will be legal in $P(\bar{r}) \setminus (X_1 \cup \dots \cup \text{int } X_{i-1} \cup \text{int } Y_1 \cup \dots \cup \text{int } Y_{i-1})$ as well.

Our procedure is now divided into 3 stages. At stage 1, we match the strata $X_{S,A,p}$ with $X_{S,A}$, for all $A \subset S \subseteq \text{act } \bar{r}$, such that $p \notin S$. It follows from the commutativity of the diagram (5.8) that each pair of simplicial subcomplexes $(X_{S,A,p}, X_{S,A})$ is isomorphic to the pair $(B_p(\bar{r}_{S,A}), P(\bar{r}_{S,A}))$. We have $|\bar{r}_{S,A}| \leq |\bar{r}| - |S| < |\bar{r}|$, hence by induction assumption, this pair can be collapsed. As a collapsing order we choose any order which does not decrease the cardinality of the set A . Take σ such that $\text{int } \sigma \subseteq \text{int } X_{S,A,p} \cup \text{int } X_{S,A}$. By Proposition (6.9) this means that $\sigma = ((W_0, T), (S \setminus A, A), \dots)$, where either $T = \emptyset$, or $T = \{p\}$. Take $\tau \supset \sigma$, such that $\dim \tau = \dim \sigma + 1$. Then by Proposition 3.2(b) there exists $q \in A(\tau)$, such that $\sigma = \Gamma_q(\tau)$. A case-by-case analysis of the ghosting construction shows that $\text{int } \tau \subseteq \text{int } X$, where X is one of the following strata: $X_{S,A}$, $X_{S,A,p}$, X_q , $X_{q,\emptyset,p}$, $X_{S,A \setminus \{q\}}$, $X_{S,A \setminus \{q\},p}$. Since the order in which we do collapses does not decrease the cardinality of A , the interiors of the last 4 of these strata have already been removed, hence the condition (6.1) is satisfied.

At stage 2, we match X_S with $X_{S,S \setminus \{p\}}$, for all $S \subseteq \text{act } \bar{r}$, such that $p \in S$, $|S| \geq 2$. By commutativity of the diagram (5.9), the pair $(X_{S,S \setminus \{p\}}, X_S)$ is isomorphic to $(B_{S \setminus \{p\}}(\bar{r}_S), P(\bar{r}_S))$. This big collapse can easily be expressed as a sequence of elementary collapses, though in a non-canonical way. For this, we pick any $q \in S \setminus \{p\}$. It exists, since we assumed that $|S| \geq 2$. Then we match pairs $(X_{S,A \cup \{q\}}, X_{S,A})$, for all $A \subseteq S \setminus \{q\}$. Again, by commutativity of the diagram (5.9), this pair is isomorphic to $(B_q(\bar{r}_{S,A}), P(\bar{r}_{S,A}))$. The order in which we arrange S does not matter for the collapsing order. Once S is fixed, the collapsing order

inside does not decrease the cardinality of A . As above, take σ such that $\text{int } \sigma \subseteq \text{int } X_{S, A \cup \{q\}} \cup \text{int } X_{S, A}$, take $\tau \supset \sigma$, such that $\dim \tau = \dim \sigma + 1$, and take $r \in A(\tau)$, such that $\sigma = \Gamma_q(\tau)$. By Proposition (6.9) we have $\sigma = ((W_0, \emptyset), (S \setminus A, A), \dots)$, or $\sigma = ((W_0, \emptyset), (S \setminus (A \cup \{q\}), A \cup \{q\}), \dots)$. Note, that both q and r are different from p , but we may have $q = r$. Again, a case-by-case analysis of the ghosting construction shows that $\text{int } \tau \subseteq \text{int } X$, where X is one of the following strata: $X_{S, A}$, $X_{S, A \cup \{q\}}$, $X_{S, A \setminus \{r\}}$, $X_{S, A \cup \{q\} \setminus \{r\}}$, X_q , X_r . Again, since collapsing order does not decrease the cardinality of A , the condition (6.1) is satisfied.

At stage 3, we collapse the pair $(X_{p,p}, X_p)$. Let us be specific. First, by Corollary 5.5 we know that $X_{p,p} = \bigcup_{\{p\} \subset S \subseteq \text{act } \bar{r}} X_{S,p}$, and it follows from Proposition prop:6.9 that $\text{int } X_{p,p} = \bigcup_{\{p\} \subset S \subseteq \text{act } \bar{r}} \text{int } X_{S,p}$. By commutativity of the diagram (5.9), the pair $(X_{p,p}, X_p)$ is isomorphic to $(B_p(\bar{r}_p), P(\bar{r}_p))$, hence it can be collapsed using the induction assumption. Clearly, the entire procedure exhausts the set Σ , and we arrive at the simplicial complex $(\partial P(\bar{r})) \setminus \text{int } B_p(\bar{r})$. \square

Corollary 6.12. *For an arbitrary round counter \bar{r} , the simplicial complex $P(\bar{r})$ is collapsible.*

Proof. Iterative use of Lemma 6.11. \square

6.2. Homeomorphic gluing.

Definition 6.13. *We say that a simplicial complex K is **simplicially homeomorphic** to a simplex Δ^A , where A is some finite set, if there exists a homeomorphism $\varphi : \Delta^A \rightarrow K$, such that for every simplex $\sigma \in \Delta^A$, the image $\varphi(\sigma)$ is a subcomplex of K .*

When we say that a CW complex is *finite* we shall mean that it has finitely many cells.

Definition 6.14. *Let X and Y be finite CW complexes. A **homeomorphic gluing data** between X and Y consists of the following:*

- a family $(A_i)_{i=1}^t$ of CW subcomplexes of X , such that $X = \bigcup_{i=1}^t A_i$,
- a family $(B_i)_{i=1}^t$ of CW subcomplexes of Y , such that $Y = \bigcup_{i=1}^t B_i$,
- a family of homeomorphisms $(\varphi_i)_{i=1}^t$, $\varphi_i : A_i \rightarrow B_i$,

satisfying the compatibility condition: if $x \in A_i \cap A_j$, then $\varphi_i(x) = \varphi_j(x)$.

Given finite CW complexes X and Y , together with homeomorphic gluing data $(A_i, B_i, \varphi_i)_{i=1}^t$ from X to Y , we define $\varphi : X \rightarrow Y$, by setting $\varphi(x) := \varphi_i(x)$, whenever $x \in A_i$. The compatibility condition from Definition 6.14 implies that $\varphi(x)$ is independent of the choice of i , hence the map $\varphi : X \rightarrow Y$ is well-defined.

Lemma 6.15. (Homeomorphism Gluing Lemma).

Assume we are given finite CW complexes X and Y , and homeomorphic gluing data $(A_i, B_i, \varphi_i)_{i=1}^t$, satisfying an additional condition:

$$(6.2) \quad \text{if } \varphi(x) \in B_i, \text{ then } x \in A_i,$$

then the map $\varphi : X \rightarrow Y$ is a homeomorphism.

Proof. First it is easy to see that φ is surjective. Take an arbitrary $y \in Y$, then there exists i such that $y \in B_i$. Take $x = \varphi_i^{-1}(y)$, clearly $\varphi(x) = y$.

Let us now check the injectivity of φ . Take $x_1, x_2 \in X$ such that $\varphi(x_1) = \varphi(x_2)$. There exists i such that $x_1 \in A_i$. Then $\varphi(x_1) = \varphi_i(x_1) \in B_i$, hence $\varphi(x_2) \in B_i$.

Condition (6.2) implies that $x_2 \in A_i$. The fact that $x_1 = x_2$ now follows from the injectivity of φ_i .

We have verified that φ is bijective, so $\varphi^{-1} : Y \rightarrow X$ is a well-defined map. We shall now prove that φ^{-1} is continuous by showing that φ takes closed sets to closed sets. To start with, let us recall the following basic property of the topology of CW complexes: a subset A of a CW complex X is closed if and only if its intersection with the closure of each cell in X is closed. Sometimes, one uses the terminology *weak topology* of the CW complex. This property was an integral part of the original J.H.C. Whitehead definition of CW complexes, see, e.g., [Hat02, Proposition A.2.] for further details.

Let us return to our situation. We claim that $A \subseteq X$ is closed, if and only if $A \cap A_i$ is closed in A_i , for each $i = 1, \dots, t$. Note first that since A_i is itself closed, a subset $S \subseteq A_i$ is closed in X if and only if it is closed in A_i , so we will skip mentioning where the sets are closed. Clearly, if A is closed, then $A \cap A_i$ is closed for all $i = 1, \dots, t$. On the other hand, assume $A \cap A_i$ is closed for all i . Let σ be a closed cell of X , we need to show that $A \cap \sigma$ is closed. Since $X = \bigcup_{i=1}^t A_i$, and A_i 's are CW subcomplexes of X , there exists i , such that $\sigma \subseteq A_i$. Then $A \cap \sigma = A \cap (A_i \cap \sigma) = (A \cap A_i) \cap \sigma$, but $(A \cap A_i) \cap \sigma$ is closed since $A \cap A_i$ is closed. Hence $A \cap \sigma$ is closed and our argument is finished. Similarly, we can show that $B \subseteq X$ is closed, if and only if $B \cap B_i$ is closed, for each $i = 1, \dots, t$.

Pick now a closed set $A \subseteq X$, we want to show that $\varphi(A)$ is closed. To start with, for all i the set $A \cap A_i$ is closed, hence $\varphi_i(A \cap A_i) \subseteq B_i$ is also closed, since φ_i is a homeomorphism. Let us verify that for all i we have

$$(6.3) \quad \varphi_i(A \cap A_i) = \varphi(A) \cap B_i.$$

Assume $y \in \varphi_i(A \cap A_i)$. On one hand $y \in \varphi_i(A_i)$, so $y \in B_i$, on the other hand, $y = \varphi_i(x)$, for $x \in A$, so $y \in \varphi(A)$. Reversely, assume $y \in \varphi(A)$ and $y \in B_i$. Then $y = \varphi(x) \in B_i$, so condition (6.2) implies that $x \in A_i$, hence $y \in \varphi(A \cap A_i)$, which proves (6.3). It follows that $\varphi(A) \cap B_i$ is closed for all i , hence $\varphi(A)$ itself is closed. This proves that φ^{-1} is continuous.

We have now shown that $\varphi^{-1} : Y \rightarrow X$ is a continuous bijection. Since X and Y are both finite CW complexes, they are compact Hausdorff when viewed as topological spaces. It is a basic fact of set-theoretic topology that a continuous bijection between compact Hausdorff topological spaces is automatically a homeomorphism, see e.g., [Mun, Theorem 26.6]. \square

The following variations of the Homeomorphism Gluing Lemma 6.15 will be useful for us.

Corollary 6.16. *Assume we are given finite CW complexes X and Y , and homeomorphic gluing data $(A_i, B_i, \varphi_i)_{i=1}^t$, satisfying an additional condition:*

$$(6.4) \quad \text{for all } I \subseteq [t] : \varphi : A_I \rightarrow B_I \text{ is a bijection.}$$

Then the map $\varphi : X \rightarrow Y$ is a homeomorphism.

Proof. Clearly, we just need to show that the condition (6.4) implies the condition (6.2). Assume $y = \varphi(x)$, $y \in B_i$, and $x \notin A_i$. Let I be the maximal set such that $y \in B_I$. The condition (6.4) implies that there exists a unique element $\tilde{x} \in A_I$, such that $\varphi(\tilde{x}) = y$. In particular, $\tilde{x} \in A_i$, hence $x \neq \tilde{x}$. Even stronger, if $x \in A_i$, for some $i \in I$, then $x, \tilde{x} \in A_i$, hence $x = \tilde{x}$, since φ_i is injective. So $x_i \notin A_i$, for

all $i \in I$. Hence, there exists $j \notin I$, such that $x \in A_j$, which implies $\varphi(x) \in B_j$, yielding a contradiction to the maximality of the set I . \square

Corollary 6.17. *Assume we are given CW complexes X and Y , a collection $(A_i)_{i=1}^t$ of CW subcomplexes of X , a collection $(B_i)_{i=1}^t$ of CW subcomplexes of Y , and a collection $(\varphi_I)_{I \subseteq [t]}$ of maps such that*

- $X = \cup_{i=1}^t A_i$, $Y = \cup_{i=1}^t B_i$;
- for every $I \subseteq [t]$, the map $\varphi_I : A_I \rightarrow B_I$ is a homeomorphism;
- for every $J \supseteq I$ the following diagram commutes

$$(6.5) \quad \begin{array}{ccc} A_J & \xrightarrow[\cong]{\varphi_J} & B_J \\ \downarrow & & \downarrow \\ A_I & \xrightarrow[\cong]{\varphi_I} & B_I \end{array}$$

Then $(A_i, B_i, \varphi_i)_{i=1}^t$ is a homeomorphic gluing data, and the map $\varphi : X \rightarrow Y$ defined by this data is a homeomorphism.

Proof. For arbitrary $1 \leq i, j \leq t$, commutativity of (6.5) implies that also the following diagram is commutative

$$\begin{array}{ccccc} A_i & \longleftarrow & A_{\{i,j\}} & \longrightarrow & A_j \\ \cong \downarrow \varphi_{\{i\}} & & \cong \downarrow \varphi_{\{i,j\}} & & \cong \downarrow \varphi_{\{j\}} \\ B_i & \longleftarrow & B_{\{i,j\}} & \longrightarrow & B_j \end{array}$$

In other words, for any $x \in A_i \cap A_j$, we have $\varphi_{\{i\}}(x) = \varphi_{\{i,j\}}(x) = \varphi_{\{j\}}(x)$. It follows that $(A_i, B_i, \varphi_{\{i\}})_{i=1}^t$ is a homomorphic gluing data. Since for all $I \subseteq [t]$, the map φ_I is a homeomorphism, it is in particular bijective, so conditions of Corollary 6.16 are satisfied, and the defined map φ is a homeomorphism. \square

6.3. Main Theorem. The fact that the protocol complexes in the immediate snapshot read/write shared memory model are homeomorphic to simplices has been folklore knowledge in the theoretical distributed computing community, [Her]. The next theorem provides a rigorous mathematical proof of this fact.

Theorem 6.18. *For every round counter \bar{r} there exists a homeomorphism*

$$\Phi(\bar{r}) : P(\bar{r}) \xrightarrow{\cong} P(\chi(\bar{r})),$$

such that

- (1) for all $V \subset \text{supp } \bar{r}$ the following diagram commutes:

$$(6.6) \quad \begin{array}{ccccc} P(\bar{r} \setminus V) & \xleftarrow{\delta_V(\bar{r})} & B_V(\bar{r}) & \xrightarrow{\beta_V(\bar{r})} & P(\bar{r}) \\ \Phi(\bar{r} \setminus V) \downarrow \cong & & & & \cong \downarrow \Phi(\bar{r}) \\ P(\chi(\bar{r}) \setminus V) & \xleftarrow{\delta_V(\chi(\bar{r}))} & B_V(\chi(\bar{r})) & \xrightarrow{\beta_V(\chi(\bar{r}))} & P(\chi(\bar{r})) \end{array}$$

(2) for all $S \subseteq \text{act } \bar{r}$ the following diagram commutes:
(6.7)

$$\begin{array}{ccccccc}
X_S(\bar{r}) & \xrightarrow{\gamma_S(\bar{r})} & P(\bar{r}_S) & \xrightarrow[\cong]{\Phi(\bar{r}_S)} & P(\chi(\bar{r}_S)) & \xrightarrow[\cong]{\tau} & P(\chi(\bar{r})_S) \xrightarrow[\gamma_S(\chi(\bar{r}))]{\gamma_S(\chi(\bar{r}))} X_S(\chi(\bar{r})) \\
& \searrow \alpha_S(\bar{r}) & & & & & \swarrow \alpha_S(\chi(\bar{r})) \\
& & P(\bar{r}) & \xrightarrow[\cong]{\Phi(\bar{r})} & P(\chi(\bar{r})) & &
\end{array}$$

where $\tau = \tau(\chi(\bar{r}_S), \chi(\bar{r})_S)$.

In particular, the complex $P(\bar{r})$ is simplicially homomorphic to $\Delta^{\text{supp } \bar{r}}$.

Proof. Our proof is a double induction, first on $|\text{supp } \bar{r}|$, then, once $|\text{supp } \bar{r}|$ is fixed, on the cardinality of the round counter \bar{r} . As a base of the induction, we note that the case $|\text{supp } \bar{r}| = 1$ is trivial, since the involved spaces are points. Furthermore, if $|\text{supp } \bar{r}|$ is fixed, and $|\bar{r}| = 0$, we take $\Phi(\bar{r})$ to be the identity map. In this case the simplicial complexes $P(\bar{r})$ and $P(\chi(\bar{r}))$ are simplices. The diagram (6.6) commutes, since also $\Phi(\bar{r} \setminus V)$ is the identity map. The condition (2) of the theorem is void, since $\text{act } \bar{r} = \emptyset$. As a matter of fact, more generally, $\Phi(\bar{r})$ can be taken to be the identity map whenever $\bar{r} = \chi(\bar{r})$, that is whenever $\bar{r}(i) \in \{0, 1\}$, for all $i \in \text{supp } \bar{r}$.

We now proceed to prove the induction step, assuming that $|\bar{r}| \geq 1$. For every pair of sets $A \subseteq S \subseteq \text{act } \bar{r}$, such that $S \neq \emptyset$, we define a map

$$\varphi_{S,A}(\bar{r}) : X_{S,A}(\bar{r}) \longrightarrow X_{S,A}(\chi(\bar{r})),$$

as follows

$$(6.8) \quad \varphi_{S,A}(\bar{r}) : X_{S,A}(\bar{r}) \xrightarrow{\gamma_{S,A}(\bar{r})} P(\bar{r}_{S,A}) \xrightarrow[\cong]{\Phi(\bar{r}_{S,A})} P(\chi(\bar{r}_{S,A})) \xrightarrow[\cong]{\tau} P(\chi(\bar{r})_{S,A}) \xrightarrow[\gamma_{S,A}(\chi(\bar{r}))]{\gamma_{S,A}(\chi(\bar{r}))} X_{S,A}(\chi(\bar{r})),$$

where $\tau = \tau(\chi(\bar{r}_{S,A}), \chi(\bar{r})_{S,A})$. Since $|\bar{r}_{S,A}| \leq |\bar{r}_S| = |\bar{r}| - |S| < |\bar{r}|$, the map $\Phi(\bar{r}_{S,A})$ is already defined by induction, so $\varphi_{S,A}(\bar{r})$ is well-defined by the sequence (6.8). Obviously, the map $\varphi_{S,A}$ is a homeomorphism for all pairs S, A .

We want to use Corollary 6.17 to construct the global homeomorphism $\Phi(\bar{r})$ by gluing the local ones $\varphi_{S,A}(\bar{r})$. In our setting here, the notations of Corollary 6.17 translate to $X = P(\bar{r})$, $Y = P(\chi(\bar{r}))$, A_I 's are $X_{S,A}(\bar{r})$'s, B_I 's are $X_{S,A}(\chi(\bar{r}))$'s, and φ_I 's are $\varphi_{S,A}(\bar{r})$'s. To satisfy the conditions of Corollary 6.17, we need to check that the following diagram commutes whenever $X_{S,A} \subseteq X_{T,B}$

$$(6.9) \quad \begin{array}{ccc} X_{S,A}(\bar{r}) & \xrightarrow[\cong]{\varphi_{S,A}(\bar{r})} & X_{S,A}(\chi(\bar{r})) \\ \downarrow i & & \downarrow j \\ X_{T,B}(\bar{r}) & \xrightarrow[\cong]{\varphi_{T,B}(\bar{r})} & X_{T,B}(\chi(\bar{r})), \end{array}$$

where i and j denote the inclusion maps.

Note, that by Proposition 5.6, we have $X_{S,A} \subseteq X_{T,B}$ if and only if either $S = T$ and $B \subseteq A$, or $T \subseteq A$. Consider first the case $S = T$, $B = \emptyset$. Consider the diagram on Figure 6.1. The leftmost pentagon is the diagram (5.9), which commutes by Proposition 5.15. The following hexagon is the diagram (6.6), where \bar{r} is replaced with \bar{r}_S . Since $|\bar{r}_S| = |\bar{r}| - |S| < |\bar{r}|$, this diagram commutes by induction. The next hexagon is the diagram (5.12), for $\chi_{C_1, D_1} = \chi(\bar{r}_S)$, $\chi_{C_2, D_2} = \chi(\bar{r})_S$, and we use the fact that $\chi(\bar{r}_S) \setminus A = \chi(\bar{r}_{S,A})$. Finally, the rightmost pentagon is also the commuting diagram (5.9), where \bar{r} is replaced with $\chi(\bar{r})$. Since removing the 3

$$\begin{array}{ccccccc}
 & & P(\bar{r}_{S,A}) & \xrightarrow[\cong]{\Phi} & P(\chi(\bar{r}_{S,A})) & \xrightarrow[\cong]{\tau} & P(\chi(\bar{r})_{S,A}) \\
 & \nearrow \gamma & \uparrow \delta & & \uparrow \delta & & \nwarrow \gamma \\
 X_{S,A}(\bar{r}) & & B_A(\bar{r}_S) & & B_A(\chi(\bar{r}_S)) & & B_A(\chi(\bar{r})_S) & & X_{S,A}(\chi(\bar{r})) \\
 \downarrow & & \downarrow \beta & & \downarrow \beta & & \downarrow \beta & & \downarrow \\
 X_S(\bar{r}) & \xrightarrow[\gamma]{} & P(\bar{r}_S) & \xrightarrow[\cong]{\Phi} & P(\chi(\bar{r}_S)) & \xrightarrow[\cong]{\tau} & P(\chi(\bar{r})_S) & \xleftarrow[\gamma]{} & X_S(\chi(\bar{r}))
 \end{array}$$

FIGURE 6.1.

inner terms of the diagram on Figure 6.1 yields the diagram (6.9) with $S = T$, $B = \emptyset$, we conclude that (6.9) commutes in this special case.

Consider now the case $S = T$, $B \subseteq A$. We have inclusions $X_{S,A} \hookrightarrow X_{S,B} \hookrightarrow X_S$, and it is easy to see that the commutativity of the diagram (6.9) for the inclusion $X_{S,A} \hookrightarrow X_{S,B}$ follows from the commutativity of the diagrams (6.9) for the inclusions $X_{S,A} \hookrightarrow X_S$ and $X_{S,B} \hookrightarrow X_S$. Hence we are done with the proof of this case.

Let us now prove the commutativity of the diagram (6.9) for the inclusion $X_{S,A} \hookrightarrow X_{T,B}$, when $T \subseteq A$. Assume first that $A = T = B \neq \emptyset$, and consider the diagram on Figure 6.2.

$$\begin{array}{ccccccc}
 & & P(\bar{r}_{S,A}) & \xrightarrow[\cong]{\Phi} & P(\chi(\bar{r}_{S,A})) & \xrightarrow[\cong]{\tau} & P(\chi(\bar{r})_{S,A}) \\
 & \nearrow \gamma & \uparrow \gamma & & \uparrow \gamma & & \nwarrow \gamma \\
 X_{S,A}(\bar{r}) & & X_{\tilde{S}}(\bar{r} \setminus A) & & X_{\tilde{S}}(\chi(\bar{r} \setminus A)) & & X_{S,A}(\chi(\bar{r})) \\
 \downarrow & & \downarrow \alpha & & \downarrow \alpha & & \downarrow \\
 X_{A,A}(\bar{r}) & \xrightarrow[\gamma]{} & P(\bar{r} \setminus A) & \xrightarrow[\cong]{\Phi} & P(\chi(\bar{r} \setminus A)) & \xleftarrow[\gamma]{} & X_{A,A}(\chi(\bar{r})),
 \end{array}$$

FIGURE 6.2.

where $\tilde{S} = S \setminus A$. A few of the maps in the diagram on Figure 6.2 need to be articulated. To start with, we have the identity $\bar{r}_{A,A} = \bar{r} \setminus A$, explaining the simplicial isomorphism $\gamma_{A,A}(\bar{r}) : X_{A,A}(\bar{r}) \rightarrow P(\bar{r} \setminus A)$. Similarly, $\chi(\bar{r})_{A,A} = \chi(\bar{r} \setminus A)$ explains $\gamma_{A,A}(\chi(\bar{r})) : X_{A,A}(\chi(\bar{r})) \rightarrow P(\chi(\bar{r} \setminus A))$. Furthermore, by (2.2) we have $\bar{r} \setminus A \downarrow \tilde{S} = \bar{r} \downarrow S \setminus A = \bar{r}_{S,A}$ and $\chi(\bar{r} \setminus A) \downarrow \tilde{S} = \chi(\bar{r}) \setminus A \downarrow \tilde{S} = \chi(\bar{r})_{S,A}$. These identities explain the presence of the maps $\gamma_{\tilde{S}}(\bar{r} \setminus A) : X_{\tilde{S}}(\bar{r} \setminus A) \rightarrow P(\bar{r}_{S,A})$, and $\gamma_{\tilde{S}}(\chi(\bar{r} \setminus A)) : X_{\tilde{S}}(\chi(\bar{r} \setminus A)) \rightarrow P(\chi(\bar{r})_{S,A})$.

Let us look at the commutativity of the diagram on Figure 6.2. The middle heptagon is the diagram (6.7) with $\bar{r} \setminus A$ instead of \bar{r} and \tilde{S} instead of S ; where we again use the identity $\bar{r} \setminus A \downarrow \tilde{S} = \bar{r} \downarrow S \setminus A$. Since $|\text{supp}(\bar{r} \setminus A)| = |\text{supp} \bar{r}| - |A| < |\text{supp} \bar{r}|$, the induction hypothesis implies that this heptagon commutes. The leftmost pentagon is (5.1), with \tilde{S} instead of S , whereas the rightmost pentagon is (5.1) as well, this time with \tilde{S} instead of S , and $\chi(\bar{r})$ instead of \bar{r} . They both commute by Proposition 5.4. Again, removing the 2 inner terms from the diagram on Figure 6.2 will yield the diagram (6.9) with $A = T = B$, so we conclude that (6.9) commutes in this special case.

In general, when $T \subseteq A$, we have a sequence of inclusions $X_{S,A} \hookrightarrow X_{S,T} \hookrightarrow X_{T,T} \hookrightarrow X_{T,B}$. Again, it is easy to see that the commutativity of the diagram (6.9) for the inclusion $X_{S,A} \hookrightarrow X_{T,B}$ follows from the commutativity of the diagrams (6.9) for the inclusions $X_{S,A} \hookrightarrow X_{S,T}$, $X_{S,T} \hookrightarrow X_{T,T}$, and $X_{T,T} \hookrightarrow X_{T,B}$. Hence we are done with the proof of this case as well.

We now know that $\Phi(\bar{r})$ is a well-defined homeomorphism between $P(\bar{r})$ and $P(\chi(\bar{r}))$. To finish the proof of the main theorem, we need to check the commutativity of the diagrams (6.6) and (6.7). The commutativity of (6.7) is an immediate consequence of (6.8), and the way $\Phi(\bar{r})$ was defined. To show that (6.6) commutes, pick any $S \subseteq \text{act } \bar{r}$, which is disjoint from A , and consider the diagram on Figure 6.3. The maps φ and ψ are as in Proposition 5.14, and the maps ρ and ν are

$$\begin{array}{ccccc}
 P(\chi(\bar{r})) & \xleftarrow{\alpha} & X_S(\chi(\bar{r})) & \xrightarrow[\cong]{\rho} & P(\chi(\bar{r}_S)) \\
 \uparrow \beta & \nwarrow \cong & \uparrow \beta & \nearrow \cong & \uparrow \beta \\
 B_V(\chi(\bar{r})) & & B_V(\bar{r}) & \xleftarrow{\alpha} & X_S(\bar{r}) & \xrightarrow{\sim} & P(\bar{r}_S) & & B_V(\bar{r}_S) \\
 \downarrow \delta & & \downarrow \delta & & \downarrow \delta & & \downarrow \delta & & \downarrow \delta \\
 P(\bar{r} \setminus V) & \xleftarrow{\alpha} & X_S(\bar{r} \setminus V) & \xrightarrow{\sim} & P(\bar{r}_{S,V}) & \xrightarrow[\cong]{\Phi} & P(\chi(\bar{r}_{S,V})) \\
 \uparrow \beta & \nwarrow \cong & \uparrow \beta & \nearrow \cong & \uparrow \beta & & \uparrow \beta \\
 B_V(\chi(\bar{r} \setminus V)) & & B_V(\bar{r} \setminus V) & \xleftarrow{\alpha} & X_S(\bar{r} \setminus V) & \xrightarrow{\sim} & P(\bar{r}_{S,V}) & & B_V(\bar{r}_{S,V}) \\
 \downarrow \delta & & \downarrow \delta & & \downarrow \delta & & \downarrow \delta & & \downarrow \delta \\
 P(\chi(\bar{r} \setminus V)) & \xleftarrow{\alpha} & X_S(\chi(\bar{r} \setminus V)) & \xrightarrow[\cong]{\nu} & P(\chi(\bar{r}_{S,V}))
 \end{array}$$

FIGURE 6.3.

given by

$$\rho : X_S(\chi(\bar{r})) \xrightarrow{\gamma_S(\chi(\bar{r}))} P(\chi(\bar{r}_S)) \xrightarrow{\tau(\chi(\bar{r}_S), \chi(\bar{r})_S)^{-1}} P(\chi(\bar{r}_S))$$

and

$$\nu : X_S(\chi(\bar{r} \setminus A)) \xrightarrow{\gamma_S(\chi(\bar{r} \setminus A))} P(\chi(\bar{r})_{S,A}) \xrightarrow{\tau(\chi(\bar{r}_{S,A}), \chi(\bar{r})_{S,A})^{-1}} P(\chi(\bar{r}_{S,A})),$$

where we use the identities $\chi(\bar{r} \setminus A) = \chi(\bar{r}) \setminus A$, $\chi(\bar{r}_S) \setminus A = \chi(\bar{r}_{S,A})$, and $\chi(\bar{r} \setminus A)_S = \chi(\bar{r})_{S,A}$, with the latter one relying on the fact that $S \cup A = \emptyset$.

Let us investigate the diagram on Figure 6.3 in some detail. The middle part is precisely the diagram (5.8), which commutes by Proposition 5.14. We have 4 hexagons surround that middle part. The hexagon on the left is the diagram (6.6) itself. The hexagon above is precisely the diagram (6.7), so it commutes. The hexagon below is the diagram (6.7) with $\bar{r} \setminus A$ instead of \bar{r} , where we use (2.3) again. This diagram commutes by the induction hypothesis. The hexagon on the right is the diagram (6.6) with \bar{r}_S instead of \bar{r} . Since $|\bar{r}_S| < |\bar{r}|$, it also commutes by the induction assumption.

Let us now show that the diagram obtained from the one on Figure 6.3 by the removal of the 9 inner terms commutes. This diagram can be factorized as shown on Figure 6.4. The left part of the diagram on Figure 6.4 is (5.8) with $\chi(\bar{r})$ instead

$$\begin{array}{ccccc}
 P(\chi(\bar{r})) & \xleftarrow{\alpha} & X_S(\chi(\bar{r})) & \rightsquigarrow^{\gamma} & P(\chi(\bar{r})_S) \xleftarrow[\cong]{\tau} P(\chi(\bar{r}_S)) \\
 \uparrow \beta & & & & \uparrow \beta \\
 B_V(\chi(\bar{r})) & & & & B_V(\chi(\bar{r})_S) \\
 \downarrow \delta & & & & \downarrow \delta \\
 P(\chi(\bar{r} \setminus V)) & \xleftarrow{\alpha} & X_S(\chi(\bar{r} \setminus V)) & \rightsquigarrow^{\gamma} & P(\chi(\bar{r})_{S,V}) \xleftarrow[\cong]{\tau} P(\chi(\bar{r}_S)_V) \\
 & & & & \uparrow \beta \\
 & & & & B_V(\chi(\bar{r}_S)) \\
 & & & & \downarrow \delta
 \end{array}$$

FIGURE 6.4.

of \bar{r} , whereas the right part of the diagram on Figure 6.4 is the diagram (5.12) with $\chi_{C_1, D_1} = \chi(\bar{r}_S)$, $\chi_{C_2, D_2} = \chi(\bar{r})_S$. They both commute, hence so does the whole diagram.

Consider now two sequences of maps in the diagram on Figure 6.3:

$$(6.10) \quad B_V(\bar{r}) \cap X_S(\bar{r}) \hookrightarrow B_V(\bar{r}) \xrightarrow{\beta} P(\bar{r}) \xrightarrow[\cong]{\Phi} P(\chi(\bar{r}))$$

and

$$(6.11) \quad X_{S, \emptyset, V}(\bar{r}) \hookrightarrow B_V(\bar{r}) \rightsquigarrow^{\delta} P(\bar{r} \setminus V) \xrightarrow[\cong]{\Phi} P(\chi(\bar{r} \setminus V)) \rightsquigarrow^{\delta} B_V(\chi(\bar{r})) \xrightarrow{\beta} P(\chi(\bar{r}))$$

It follows by a simple diagram chase that the commutativities in the diagram on Figure 6.3 which we have shown imply the equality of these two maps. This is true for all S , such that $S \subseteq \text{act } \bar{r}$ and $S \cap V = \emptyset$. On the other hand, the subcomplexes $X_{S, \emptyset, V}(\bar{r})$, where $S \subseteq \text{act } \bar{r}$, $S \cap V = \emptyset$, cover $B_V(\bar{r})$. As a matter of fact, the simplicial isomorphisms ψ and $\delta_V(\bar{r})$ show that they induce a stratification which is isomorphic to the stratification of $P(\bar{r} \setminus V)$ by $X_S(\bar{r} \setminus V)$. The fact that they cover $B_V(\bar{r})$ completely implies that the maps (6.10) and (6.11) remain the same after the first term is skipped, which is the same as to say that (6.6) commutes. This concludes the proof. \square

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